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The Lower Fourier Dimensions of In-Homogeneous Self-similar Measures

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Abstract

The in-homogeneous self-similar measure μ is defined by the relation

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + p\nu,$$

where $(p_1, \ldots, p_N, p)$ is a probability vector, each $S_j : \mathbb{R}^d \to \mathbb{R}^d$, $j = 1, \ldots, N$, is a contraction similarity, and ν is a Borel probability measure on $\mathbb{R}^d$ with compact support. In this paper, we study the asymptotic behavior of the Fourier transforms of in-homogeneous self-similar measures. We obtain non-trivial lower and upper bounds for the qth lower Fourier dimensions of the in-homogeneous self-similar measures without any separation conditions. Moreover, if the IFS satisfies some separation conditions, the lower bounds for the qth lower Fourier dimensions can be improved. These results confirm conjecture 2.5 and give a positive answer to the question 2.7 in Olsen and Snigireva's paper (Math Proc Camb Philos Soc 144(2):465–493, 2008).

Keywords In-homogeneous self-similar measure · Fourier transform · Separation condition

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1 Introduction and Statement of Results

Let $\mathbf{I} = \{S_1, \ldots, S_N\}$ be a family of contracting similarities on $\mathbb{R}^d$. It is a fundamental result in fractal geometry that there exists a unique, non-empty compact set $K_\emptyset \subset \mathbb{R}^d$ such that

$$K_\emptyset = \bigcup_{j=1}^{N} S_j(K_\emptyset), \tag{1.1}$$

see Hutchinson [11]. We call $\mathbf{I} = \{S_j\}_{j=1}^N$ an *iterated function system* (IFS) of similarities and $K_\emptyset$ the *self-similar set* generated by $\mathbf{I}$. In order to understand the fractal structures of self-similar sets, one studies the so-called self-similar measures. More precisely, given a probability vector $\mathbf{p} = (p_1, \ldots, p_N)$, i.e. all $p_j > 0$ and $\sum_j p_j = 1$, there exists a unique Borel probability measure μ_0 supported on $K_\emptyset$ such that

$$\mu_0 = \sum_{j=1}^{N} p_j \mu_0 \circ S_j^{-1}. \tag{1.2}$$

We say that the measure μ_0 is the *self-similar measure* generated by $(\mathbf{I}, \mathbf{p})$. Self-similar sets and measures play an important role in the study of fractal geometry and we refer the reader to [6, 7, 11] and the references therein for the detailed properties of self-similar sets and measures. Observe that the self-similar measure μ_0 can be viewed as the unique solution of the following homogeneous equation

$$\mu - \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} = 0.$$

This viewpoint suggests to us the following natural generalization of self-similar measures.

Definition 1.1 Let $\mathbf{I} = \{S_j\}_{j=1}^N$ be an IFS of similarities, let $\mathbf{p} = (p_1, \ldots, p_N, p)$ be a probability vector, and let ν be a Borel probability measure on $\mathbb{R}^d$ with compact support. A Borel probability measure μ satisfying the equation

$$\mu - \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} = p\nu \tag{1.3}$$

is called the *in-homogeneous self-similar measure* generated by $(\mathbf{I}, \mathbf{p}, \nu)$.

The existence and uniqueness of such measures is well-known, see e.g. [17, 18]. Furthermore, it has been shown that the support of the in-homogeneous self-similar measure μ is equal to the unique non-empty compact set K_C, called the *in-homogeneous self-similar set*, such that

$$K_C = \bigcup_{j=1}^{N} S_j(K_C) \cup C, \tag{1.4}$$

where C is the compact support of the measure ν.

In-homogeneous self-similar sets and measures were first introduced by Barnsley and Demko in [3], where they considered some examples of in-homogeneous self-similar measures. In [1, 2], the in-homogeneous terms ν and C are called the *condensation measure* and the *condensation set* respectively. One of the most important topics in the field of in-homogeneous self-similar measures is to relate various properties of in-homogeneous self-similar measures to the associated condensation measures. See [8] for a discussion of box dimensions of in-homogeneous self-similar measures, and see [17, 25] for L^q spectra and Rényi dimensions.

The Fourier transform of a Borel probability measure μ on $\mathbb{R}^d$ is defined by

$$\widehat{\mu}(x) := \int_{\mathbb{R}^d} e^{i\langle x,y\rangle}\, \mathrm{d}\mu(y), \quad x \in \mathbb{R}^d,$$

where $\langle x, y\rangle$ denotes the Euclidean inner product of x and y. The study of the Fourier transforms of measures has a long history. In recent years, the behavior of the Fourier transforms of measures at infinity has received much attention. Li and Sahlsten [16] gave the sufficient conditions for a self-similar measure μ to be a Rajchman measure, that is, its Fourier transform $|\widehat{\mu}(x)| \to 0$ as $|x| \to 0$. See [15] for a similar discussion of self-affine measures. In [19], Solomyak proved that almost every self-similar measures on the real line has a power decay of the Fourier transform at infinity. Varjú and Yu [24] provided quantitative decay rates of Fourier transform of some self-similar measures, using random walks on lattices and Diophantine approximation in number fields. Considering the interplay between the behavior of Fourier transform and the absolute continuity of a measure, that the faster the Fourier transform of a measure tends to zero the more regular the measure is, we focus our consideration on the Fourier dimensions of in-homogeneous self-similar measures.

Definition 1.2 For a Borel probability measure μ on $\mathbb{R}^d$, we define the *infinity lower Fourier dimension* $\underline{\Delta}_\infty(\mu)$ of the measure μ by

$$\underline{\Delta}_\infty(\mu) = \liminf_{R\to\infty} \frac{\log \sup_{|x|\geq R} |\widehat{\mu}(x)|}{-\log R}. \tag{1.5}$$

For $q \in (0, \infty)$, we define the *qth lower Fourier dimension* $\underline{\Delta}_q(\mu)$ of the measure μ by

$$\underline{\Delta}_q(\mu) = \liminf_{R\to\infty} \frac{\log\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\mu}(x)|^q\, \mathrm{d}x\right)^{\frac{1}{q}}}{-\log R}, \tag{1.6}$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure.

The Fourier dimension of a measure μ on $\mathbb{R}^d$ measures the polynomial rate of decay of Fourier transform of μ. Usually it is very difficult to calculate the Fourier dimension bounds directly. During the past 20 years, there has been an enormous interest in investigating Fourier dimensions of self-similar measures and there is a huge body of literature discussing this problem, see, for example, Bluhm [4], Hu [9], Hu and Lau [10], Lau [12], Lau and Wang [14], Strichartz [20–23], while the results are limited in the in-homogeneous case. The main purpose of this paper is to investigate the asymptotic behavior of the Fourier transform of an in-homogeneous self-similar measure and relate its qth lower Fourier dimensions to the corresponding self-similar measure and the condensation measure.

We now proceed to describe our main results in more detail.

1.1 Statement of Results

Throughout this paper, we fix an in-homogeneous self-similar measure μ satisfying

$$\mu = \sum_{j=1}^{N} p_j \mu \circ S_j^{-1} + p\nu, \tag{1.7}$$

where $(p_1, \dots, p_N, p)$ is a probability vector, ν is a Borel probability measure on $\mathbb{R}^d$ with the compact support C, and $\{S_j\}_{j=1}^N$ is an IFS of similarities. Each contraction similarity S_j has the form

$$S_j x = r_j A_j x + b_j, \tag{1.8}$$

where $0 < r_j < 1$, A_j is an orthogonal matrix, and b_j is a vector in $\mathbb{R}^d$.

Definition 1.3 We say that an IFS, $\{S_j\}_{j=1}^N$, satisfies the *open set condition* (OSC) if there exists an open, non-empty and bounded set U such that $\bigcup_j S_j(U) \subset U$ and $S_j(U) \cap S_k(U) = \emptyset$ for all $j \neq k$.

Definition 1.4 We say that an IFS, $\{S_j\}_{j=1}^N$, satisfies the *equi-contractive condition* if all the contraction ratios $r_1, \dots, r_N$ coincide, i.e. if $r_1 = \cdots = r_N$.

Main Theorem. Let μ be the in-homogeneous self-similar measure with the condensation measure ν. Define t and u by

$$\sum_j p_j r_j^{-t} = 1, \quad \sum_j p_j^2 r_j^{-u} = 1.$$

Then the following statements hold.

(i) For all $q \geq 1$, we have

$$\underline{\Delta}_q(\nu) \geq \underline{\Delta}_q(\mu) \geq \min\{\underline{\Delta}_q(\nu), t\}.$$

(ii) If $r_1 = \cdots = r_N = r,\ A_1 = \cdots = A_N = A$, then for all $q > 0$,

$$\underline{\Delta}_q(\nu) \geq \underline{\Delta}_q(\mu) \geq \min\{\underline{\Delta}_q(\nu), t\}.$$

(iii) If $r_1 = \cdots = r_N = r,\ A_1 = \cdots = A_N = A$ and the OSC is satisfied, then

$$\underline{\Delta}_q(\mu) \geq \begin{cases} \min\left\{\frac{u}{2}, \ \underline{\Delta}_q(\nu)\right\}, & \text{if} \quad 0 < q \leq 2; \\ \min\left\{\frac{u}{q}, \ \underline{\Delta}_q(\nu)\right\}, & \text{if} \quad 2 \leq q. \end{cases}$$

Recently, Zhang and Xiao studied the infinity lower Fourier dimensions $\underline{\Delta}_\infty(\mu)$ of the in-homogeneous self-similar measure μ and obtained the following result in [26].

Theorem 1.5 ([26], Theorem 7) *Let μ be the in-homogeneous self-similar measure, and let ν be the condensation measure. Then*

$$\underline{\Delta}_\infty(\nu) \geq \underline{\Delta}_\infty(\mu) \geq \min\{\underline{\Delta}_\infty(\nu), t\}.$$

In [18], it has been shown that in the equi-contrative case the lower bounds for the infinity lower Fourier dimension $\underline{\Delta}_\infty(\mu)$ and the 2nd lower Fourier dimension $\underline{\Delta}_2(\mu)$ satisfy analogous equations. It is natural to expect that the similar result holds for an arbitrary qth lower Fourier dimension $\underline{\Delta}_q(\mu)$. Thus, the authors raised the following question in [18].

Question 1.6 ([18], Question 2.7) *Assume that the OSC is satisfied, $r_1 = \cdots = r_N = r > 0$ and $A_1 = \cdots = A_N = A$, where A is a rotation matrix. Let t be defined by*

$$\sum_j p_j r^{-t} = 1 \ \ i.e. \ \ t = \frac{\log(1-p)}{\log r}.$$

Is it true that

$$\underline{\Delta}_q(\mu) \geq \min\{\underline{\Delta}_q(\nu), t\}. \tag{1.9}$$

for all $q > 0$? Is (1.9) true even if the equi-contractive condition is not satisfied?

Remark 1.7 Note that our Main Theorem (i) and (ii) give an affirmative answer to the Question 1.6. It is also worth to be stressed that in the case of $q \geq 1$, the lower bound (1.9) for $\underline{\Delta}_q(\mu)$ is true even without the open set condition, the equi-contractive condition and the assumption that $A_1 = \cdots A_N = A$.

The Fourier dimension of self-similar measures satisfying the open set condition is well understood. In particular, the 2nd Fourier dimension $\underline{\Delta}_2(\mu_0)$ of a self-similar measure μ_0 satisfying (1.2) has received much attention and has been investigated in

[13, 14, 21–23]. It is proved in [21] that if the OSC is satisfied, then the lower bound for $\underline{\Delta}_2(\mu_0)$ can be improved as follows:

$$\underline{\Delta}_2(\mu_0) \geq \frac{u}{2}.$$

Furthermore, it is also proved that if the equi-contractive condition and some further conditions are satisfied, then

$$\underline{\Delta}_2(\mu_0) = \frac{u}{2}.$$

In view of the above results, it is natural to ask for an in-homogeneous analogue and, hence, Olsen and Snigireva conjectured that the results on the 2nd Fourier dimension of the in-homogeneous self-similar measures can be improved as follows.

Conjecture 1.8 ([18], Conjecture 2.5) *Assume that the OSC is satisfied, $r_1 = \cdots = r_N = r > 0$ and $A_1 = \cdots = A_N = A$, where A is a rotation matrix. We conjecture*

$$\underline{\Delta}_2(\mu) \geq \min\left\{\frac{u}{2}, \underline{\Delta}_2(\nu)\right\}.$$

Remark 1.9 Note that our Main Theorem (iii) confirms the Conjecture 1.8.

Our proof of the Main Theorem (iii) relies heavily on the assumption that all the contractions S_j are equal up to translations, i.e., $r_1 = \cdots = r_N$ and $A_1 = \cdots = A_N$. In fact, we can also prove the similar result without this assumption. For the compensation, we require the following *condensation open set condition*, appearing in [5].

Definition 1.10 We say that an IFS, $\{S_j\}_{j=1}^N$, together with a condensation set C, satisfies the *condensation open set condition* (COSC), if the IFS satisfies the open set condition and the open set U can be chosen such that

$$C \subset U \setminus (\bigcup_j S_j\overline{U}).$$

Theorem 1.11 *Assume that the IFS together with a condensation set C satisfies the COSC. Let μ be the in-homogeneous self-similar measure with the condensation measure ν. Then*

$$\underline{\Delta}_2(\nu) \geq \underline{\Delta}_2(\mu) \geq \min\left\{\frac{u}{2}, \underline{\Delta}_2(\nu)\right\}.$$

The paper is organized as follows. Sections 2, 3 and 4 are devoted to the proof of the Main Theorem (i), (ii) and (iii) respectively. More precisely, in Sect. 2, we obtain the lower and upper bounds for the qth lower Fourier dimensions of the in-homogeneous self-similar measures, for $q \geq 1$. In Sect. 3, we study the lower bounds for $\underline{\Delta}_q(\mu)$ in the case that $0 < q < 1$ under the assumption that all the contractions S_j are equal up to translations, i.e., $r_1 = \cdots = r_N$ and $A_1 = \cdots = A_N$. In Sect. 4, the lower bounds for the qth lower Fourier dimensions are improved under the open set condition. Finally, in Sect. 5, we investigate the IFS satisfying the condensation open set condition and prove Theorem 1.11.

2 Proof of the Main Theorem (i)

In this section, we study the qth lower Fourier dimensions of the in-homogeneous self-similar measures and give the proof of the Main Theorem (i). We begin by introducing some notations that will be used throughout the paper. Let

$$\mathcal{I}^n = \{1, \ldots, N\}^n,$$
$$\mathcal{I}^* = \bigcup_n \mathcal{I}^n,$$

i.e. $\mathcal{I}^n$ is the family of all finite strings $\mathbf{j} = j_1 \ldots j_n$ of length n with entries $j_k \in \{1, \ldots, N\}$, and $\mathcal{I}^*$ denotes the family of all finite strings $\mathbf{j} = j_1 \ldots j_n$ with entries $j_k \in \{1, \ldots, N\}$. For a finite string $\mathbf{j} = j_1 \ldots j_n$, let $|\mathbf{j}|$ denote the length of $\mathbf{j}$, i.e. $|\mathbf{j}| = n$, and we write $S_\mathbf{j} = S_{j_1} \circ \cdots \circ S_{j_n}$. Then $S_\mathbf{j}$ is a contraction similarity on $\mathbb{R}^d$ and so has the form

$$S_\mathbf{j} x = r_\mathbf{j} A_\mathbf{j} x + b_\mathbf{j}, \tag{2.1}$$

where $r_\mathbf{j} = r_{j_1} \ldots r_{j_n} \in (0, 1)$, $A_\mathbf{j} = A_{j_1} \ldots A_{j_n}$ is an orthogonal matrix and $b_\mathbf{j}$ is a vector in $\mathbb{R}^d$. Similarly, we define $p_\mathbf{j} = p_{j_1} \ldots p_{j_n}$. Then it follows easily from (1.7) that

$$\mu = \sum_{|\mathbf{j}|=n} p_\mathbf{j} \mu \circ S_\mathbf{j}^{-1} + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_\mathbf{j} \nu \circ S_\mathbf{j}^{-1}. \tag{2.2}$$

Taking Fourier transforms on both sides of (1.7), we have

$$\widehat{\mu}(x) = \sum_{j=1}^{N} p_j e^{i\langle x, b_j \rangle} \widehat{\mu}(L_j x) + p\widehat{\nu}(x), \tag{2.3}$$

where $L_j = r_j A_j^*$ and A^* denote the conjugate transpose of a matrix A. Taking Fourier transforms on both sides of (2.2), putting $L_\mathbf{j} = r_\mathbf{j} A_\mathbf{j}^* = L_{j_n} \ldots L_{j_1}$, we have

$$\widehat{\mu}(x) = \sum_{|\mathbf{j}|=n} p_\mathbf{j} e^{i\langle x, b_\mathbf{j} \rangle} \widehat{\mu}(L_\mathbf{j} x) + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_\mathbf{j} e^{i\langle x, b_\mathbf{j} \rangle} \widehat{\nu}(L_\mathbf{j} x). \tag{2.4}$$

We record one obvious fact, which we will use repeatedly.

Lemma 2.1 *Let* $\{a_k\}_{k=1}^n$ *be a finite sequence of positive numbers. Then for any* $q > 0$,

$$\left(\sum_{k=1}^n a_k\right)^q \leq n^q \sum_{k=1}^n a_k^q.$$

Proof It is easy to see that

$$\left(\sum_{k=1}^n a_k\right)^q \leq n^q \max_k a_k^q \leq n^q \sum_{k=1}^n a_k^q.$$

□

We shall first study the upper bounds for the qth lower Fourier dimensions $\underline{\Delta}_q(\mu)$ for all $q > 0$.

Proposition 2.2 *Let* μ *be the in-homogeneous self-similar measure with the condensation measure* ν. *Then* $\underline{\Delta}_q(\nu) \geq \underline{\Delta}_q(\mu)$ *for all* $q > 0$.

Proof Recall that

$$\widehat{\mu}(x) = \sum_{j=1}^N p_j e^{i\langle x, b_j\rangle}\widehat{\mu}(L_j x) + p\widehat{\nu}(x).$$

Noting that $p_j \in (0, 1)$, we therefore conclude from Lemma 2.1 that

$$|\widehat{\nu}(x)|^q \leq \left(\frac{N+1}{p}\right)^q \left(|\widehat{\mu}(x)|^q + \sum_{j=1}^N |\widehat{\mu}(L_j x)|^q\right). \tag{2.5}$$

Without loss of generality, we may assume that $\underline{\Delta}_q(\mu) > 0$. Fix $\epsilon \in (0, \underline{\Delta}_q(\mu))$. It follows from the definition of $\underline{\Delta}_q(\mu)$ that there exists a constant $c > 0$ such that

$$\frac{1}{\mathcal{L}^d(B(0, R))}\int_{B(0,R)} |\widehat{\mu}(x)|^q \, \mathrm{d}x \leq cR^{-q(\underline{\Delta}_q(\mu)-\epsilon)},$$

for all $R > 0$, whence

$$\frac{1}{\mathcal{L}^d(B(0, R))}\int_{B(0,R)} |\widehat{\mu}(L_j x)|^q \, \mathrm{d}x \leq c(r_j R)^{-q(\underline{\Delta}_q(\mu)-\epsilon)}.$$

Combining these and (2.5), we conclude that

$$\frac{1}{\mathcal{L}^d(B(0, R))}\int_{B(0,R)} |\widehat{\nu}(x)|^q \, \mathrm{d}x$$

$$\le \left(\frac{N+1}{p}\right)^q \left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\mu}(x)|^q \,\mathrm{d}x + \sum_j \frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\mu}(L_j x)|^q \,\mathrm{d}x\right)$$
$$\le \left[c\left(\frac{N+1}{p}\right)^q \left(1+\sum_j r_j^{-q(\underline{\Delta}_q(\mu)-\epsilon)}\right)\right] \cdot R^{-q(\underline{\Delta}_q(\mu)-\epsilon)},$$

which clearly implies that $\underline{\Delta}_q(\nu) \ge \underline{\Delta}_q(\mu) - \epsilon$. Letting $\epsilon \to 0$, the statement follows. □

We then study the lower bounds for the qth lower Fourier dimensions $\underline{\Delta}_q(\mu)$ in the case of $q \ge 1$. Note that the open set condition, the equi-contractive condition and the assumption that $A_1 = \cdots = A_n$ play no role in this case.

Proposition 2.3 *Let μ be the in-homogeneous self-similar measure with the condensation measure ν. For all $q \ge 1$, we have $\underline{\Delta}_q(\mu) \ge \min\{\underline{\Delta}_q(\nu), t\}$.*

Proof By taking absolute value in (2.4), we see that

$$|\widehat{\mu}(x)| \le \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} |\widehat{\mu}(L_{\mathbf{j}}x)| + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} |\widehat{\nu}(L_{\mathbf{j}}x)|, \tag{2.6}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Noting that $q \ge 1$, we deduce from (2.6) and Minkowski inequality that, for all $R > 0$ and $n \in \mathbb{N}$,

$$\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\mu}(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} \left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\mu}(L_{\mathbf{j}}x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$
$$+ p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} \left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\nu}(L_{\mathbf{j}}x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}. \tag{2.7}$$

Set $r_{\min} = \min_j r_j$, $r_{\max} = \max_j r_j$ and

$$R_n = \frac{1}{r_{\min}^n}$$

throughout the remaining parts of the proof of Proposition 2.3.

In order to obtain the lower bounds for $\underline{\Delta}_q(\mu)$, we need to distinguish two cases.

Case 1 We first show that $\underline{\Delta}_q(\mu) \ge \underline{\Delta}_q(\nu)$ if $0 \le \underline{\Delta}_q(\nu) \le t$. Without loss of generality, we could assume that $\underline{\Delta}_q(\nu) > 0$. Fix $\epsilon \in (0, \underline{\Delta}_q(\nu))$. It follows from the definition of $\underline{\Delta}_q(\nu)$ that there exists a constant $c > 0$ such that

$$\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\nu}(L_{\mathbf{j}}(x))|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le c(r_{\mathbf{j}}R)^{-(\underline{\Delta}_q(\nu)-\epsilon)}, \tag{2.8}$$

for any $\mathbf{j} \in \mathcal{I}^*$ and $R > 0$. Since $\underline{\Delta}_q(\nu) \le t$, we have

$$\sum_j p_j r_j^{-(\underline{\Delta}_q(\nu)-\epsilon)} < \sum_j p_j r_j^{-t} = 1. \tag{2.9}$$

Define s by

$$\sum_j p_j r_{\min}^{-s} = 1. \tag{2.10}$$

Then it is easy to see that $\rho := s/t \in (0, 1]$. Using (2.7), (2.8), (2.9), (2.10) and the fact that $|\widehat{\mu}(x)| \le 1$, we therefore conclude that

$$\begin{aligned}
&\left(\frac{1}{\mathcal{L}^d(B(0, R_n^\rho))} \int_{B(0,R_n^\rho)} |\widehat{\mu}(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\le \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} \left(\frac{1}{\mathcal{L}^d(B(0, R_n^\rho))} \int_{B(0,R_n^\rho)} |\widehat{\nu}(L_{\mathbf{j}}x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \\
&\le r_{\min}^{sn} + cp \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} (r_{\mathbf{j}} R_n^\rho)^{-(\underline{\Delta}_q(\nu)-\epsilon)} \\
&= (R_n^\rho)^{-t} + cp \sum_{k=0}^{n-1} \left(\sum_j p_j r_j^{-(\underline{\Delta}_q(\nu)-\epsilon)}\right)^k (R_n^\rho)^{-(\underline{\Delta}_q(\nu)-\epsilon)} \\
&\le (R_n^\rho)^{-t} + cp (R_n^\rho)^{-(\underline{\Delta}_q(\nu)-2\epsilon)},
\end{aligned}$$

where, for the last inequality, we used the fact that $n \le R_n^{\rho\epsilon}$ for n large enough. This implies that $\underline{\Delta}_q(\mu) \ge \underline{\Delta}_q(\nu) - 2\epsilon$, and letting $\epsilon \to 0$ gives the desired result.

Case 2 It remains to show that $\underline{\Delta}_q(\mu) \ge t$ if $t < \underline{\Delta}_q(\nu)$. Fix $\epsilon > 0$ small enough. Then there exists a constant $c > 0$ such that

$$\left(\frac{1}{\mathcal{L}^d(B(0, R))} \int_{B(0,R)} |\widehat{\mu}(L_{\mathbf{j}}(x))|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le c(r_{\mathbf{j}} R)^{-(\underline{\Delta}_q(\mu)-\epsilon)},$$

and

$$\left(\frac{1}{\mathcal{L}^d(B(0, R))} \int_{B(0,R)} |\widehat{\nu}(L_{\mathbf{j}}(x))|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le c(r_{\mathbf{j}} R)^{-(\underline{\Delta}_q(\nu)-\epsilon)},$$

for any $\mathbf{j} \in \mathcal{I}^*$ and $R > 0$. Thus, it follows from (2.7) that for all $R > 0$ and $n \in \mathbb{N}$, one has

$$\left(\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)}|\widehat{\mu}(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le cR^{-(\underline{\Delta}_q(\mu)-\epsilon)}\left(\sum_j p_j r_j^{-(\underline{\Delta}_q(\mu)-\epsilon)}\right)^n + cpR^{-(\underline{\Delta}_q(\nu)-\epsilon)}\sum_{k=0}^{n-1}\left(\sum_j p_j r_j^{-(\underline{\Delta}_q(\nu)-\epsilon)}\right)^k. \tag{2.11}$$

Arguing by contradiction, we assume that $\underline{\Delta}_q(\mu) < t$. Thus, we can choose a constant $\epsilon_0 > 0$ such that

$$\underline{\Delta}_q(\mu) + \frac{2\log r_{\min}}{\log r_{\max}}\epsilon_0 < t.$$

Since $\sum_j p_j r_j^{-t} = 1$, we obtain that

$$\sum_j p_j r_j^{-(\underline{\Delta}_q(\mu)-\epsilon)} \le \left(\sum_j p_j r_j^{-(\underline{\Delta}_q(\mu)+\frac{2\log r_{\min}}{\log r_{\max}}\epsilon_0)}\right)\cdot\left(\frac{1}{r_{\min}}\right)^{-(\frac{\log r_{\max}}{\log r_{\min}}\epsilon+2\epsilon_0)} \le \left(\frac{1}{r_{\min}}\right)^{-(\epsilon+\epsilon_0)},$$

and

$$\sum_j p_j r_j^{-(\underline{\Delta}_q(\nu)-\epsilon)} \le \left(\sum_j p_j r_j^{-t}\right)\cdot\left(\frac{1}{r_{\min}}\right)^{\underline{\Delta}_q(\nu)-t-\epsilon} = \left(\frac{1}{r_{\min}}\right)^{\underline{\Delta}_q(\nu)-t-\epsilon},$$

provided that $\epsilon > 0$ is small enough. Combining this with (2.11), we have

$$\left(\frac{1}{\mathcal{L}^d(B(0,R_n))}\int_{B(0,R_n)}|\widehat{\mu}(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} \le cR_n^{-(\underline{\Delta}_q(\mu)-\epsilon)}\left(\sum_j p_j r_j^{-(\underline{\Delta}_q(\mu)-\epsilon)}\right)^n + cpR_n^{-(\underline{\Delta}_q(\nu)-\epsilon)}\sum_{k=0}^{n-1}\left(\sum_j p_j r_j^{-(\underline{\Delta}_q(\nu)-\epsilon)}\right)^k \le cR_n^{-(\underline{\Delta}_q(\mu)+\epsilon_0)} + \frac{cpr_{\min}^{\underline{\Delta}_q(\nu)-t-\epsilon}}{1-r_{\min}^{\underline{\Delta}_q(\nu)-t-\epsilon}}R_n^{-t},$$

provided that $\epsilon > 0$ is small enough. Thus, $\underline{\Delta}_q(\mu) \ge \underline{\Delta}_q(\mu) + \epsilon_0$, a contradiction. This completes the proof. □

***Proof of the Main Theorem* (i)** The proof of the Main Theorem (i) follows immediately from Propositions 2.2 and 2.3. □

3 Proof of the Main Theorem (ii)

In this section we prove the Main Theorem (ii) and, thus, we will assume that all the contractions S_j are equal up to translations, i.e., $r_1 = \cdots = r_N = r,\ A_1 = \cdots = A_N = A$. Under this assumption, putting $L = rA^*$, it follows from (2.4) that

$$\begin{aligned} |\widehat{\mu}(x)| &\le \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} |\widehat{\mu}(L^n x)| + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} |\widehat{\nu}(L^k x)| \\ &\le (1-p)^n + \sum_{k=0}^{n-1} (1-p)^k |\widehat{\nu}(L^k x)|, \end{aligned} \tag{3.1}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Proof We only need to consider the case that $0 < q < 1$. In this case, we deduce from (3.1) that

$$|\widehat{\mu}(x)|^q \le (1-p)^{nq} + \sum_{k=0}^{n-1} (1-p)^{kq} |\widehat{\nu}(L^k x)|^q,$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Without loss of generality, we may assume that $\underline{\Delta}_q(\nu) > 0$. Fix $\epsilon > 0$. It follows from the definition of $\underline{\Delta}_q(\nu)$ that there exists a constant $c > 0$ such that,

$$\frac{1}{\mathcal{L}^d(B(0,R))} \int_{B(0,R)} |\widehat{\nu}(L^k x)|^q \, \mathrm{d}x \le c(r^k R)^{-q(\underline{\Delta}_q(\nu)-\epsilon)},$$

for all $R > 0$ and $k \in \mathbb{N}$. Thus, we obtain that

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} |\widehat{\mu}(x)|^q \, \mathrm{d}x \\ &\le (1-p)^{nq} + \sum_{k=0}^{n-1} (1-p)^{kq} \frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} |\widehat{\nu}(L^k x)|^q \, \mathrm{d}x \\ &\le r^{nqt} + c \sum_{k=0}^{n-1} r^{kqt} r^{(n-k)q(\underline{\Delta}_q(\nu)-\epsilon)} \\ &\le r^{nqt} + c \sum_{k=0}^{n-1} \left(r^{kq\cdot\min\{t,(\underline{\Delta}_q(\nu)-\epsilon)\}} \right) \left(r^{(n-k)q\cdot\min\{t,(\underline{\Delta}_q(\nu)-\epsilon)\}} \right) \\ &\le \left(\frac{1}{r^n}\right)^{-qt} + c \left(\frac{1}{r^n}\right)^{-q\cdot\min\{t-\epsilon,(\underline{\Delta}_q(\nu)-2\epsilon)\}}, \end{aligned} \tag{3.2}$$

where, for the last inequality, we used the fact that $n \le r^{-qn\epsilon}$ for n large enough. This clearly implies that $\underline{\Delta}_q(\mu) \ge \min\{t-\epsilon, \underline{\Delta}_q(\nu) - 2\epsilon\}$. The statement follows by letting $\epsilon \to 0$, □

4 Proof of Main Theorem (iii)

In this section, we will prove the Main Theorem (iii). As in Sect. 3, we shall assume that $r_1 = \cdots = r_N = r$ and $A_1 = \cdots = A_N = A$ and set $L = rA^*$. Using this notation, it therefore follows from (2.4) that

$$\widehat{\mu}(x) = \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L^n x) + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L^k x), \tag{4.1}$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Furthermore, we also assume that the IFS, $\{S_j\}_{j=1}^N$, satisfies the open set condition.

Lemma 4.1 *Assume that the OSC is satisfied. Assume further that $r_1 = \cdots = r_N = r$ and $A_1 = \cdots = A_N = A$. Then there exists a constant $\kappa > 0$ such that*

$$|b_{\mathbf{j}_1} - b_{\mathbf{j}_2}| \ge \kappa r^n$$

for all $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{I}^n$ with $\mathbf{j}_1 \ne \mathbf{j}_2$.

Proof It follows from the open set condition that there exist an open, non-empty and bounded set U such that $S_j(U) \subset U$ for all j, and $S_j(U) \cap S_k(U) = \emptyset$ for all $j \ne k$. Fix a point $x \in U$ and write $\kappa = 2\,\mathrm{dist}(x, \partial U)$. It is easy to see that $\kappa > 0$ since U is open. Hence,

$$\begin{aligned} |b_{\mathbf{j}_1} - b_{\mathbf{j}_2}| &= |S_{\mathbf{j}_1}(x) - S_{\mathbf{j}_2}(x)| \\ &\ge \mathrm{dist}(S_{\mathbf{j}_1}(x), \partial S_{\mathbf{j}_1}(U)) + \mathrm{dist}(S_{\mathbf{j}_2}(x), \partial S_{\mathbf{j}_2}(U)) = \kappa r^n. \end{aligned}$$

This completes the proof. □

Lemma 4.2 *Assume that the OSC is satisfied. Assume further that $r_1 = \cdots = r_N = r$ and $A_1 = \cdots = A_N = A$. Let C be the condensation set. Then there exists a constant N_0 such that for any $k \in \mathbb{N}$ and $\mathbf{i} \in \mathcal{I}^k$, we have*

$$\#\left\{\mathbf{j} \in \mathcal{I}^k : \mathrm{dist}(S_{\mathbf{i}}C, S_{\mathbf{j}}C) \le r^k\right\} \le N_0.$$

Proof Fix $x_0 \in C$ and $\mathbf{i} \in \mathcal{I}^k$. By Lemma 4.1, there exists a constant $\kappa > 0$ such that for any $\mathbf{j}_1, \mathbf{j}_2 \in \mathcal{I}^k$ with $\mathbf{j}_1 \ne \mathbf{j}_2$,

$$|S_{\mathbf{j}_1}x_0 - S_{\mathbf{j}_2}x_0| = |b_{\mathbf{j}_1} - b_{\mathbf{j}_2}| \ge \kappa r^k,$$

which implies that the collection of balls $\{B(S_{\mathbf{j}}x_0, \frac{1}{3}\kappa r^k)\}_{\mathbf{j}\in\mathcal{I}^k}$ is pairwise disjoint. Moreover, if $\mathrm{dist}(S_{\mathbf{i}}C, S_{\mathbf{j}}C) \leq r^k$ for some $\mathbf{j} \in \mathcal{I}^k$, then the corresponding ball $B(S_{\mathbf{j}}x_0, \frac{1}{3}\kappa r^k)$ is contained in the ball centered at $S_{\mathbf{i}}x_0$ with radius $(1 + 2\,\mathrm{diam}\, C + \frac{1}{3}\kappa)r^k$. Summing the volume of these balls, it follows that

$$\#\left\{\mathbf{j} \in \mathcal{I}^k : \mathrm{dist}(S_{\mathbf{i}}C, S_{\mathbf{j}}C) \leq r^k\right\} \cdot \left(\frac{1}{3}\kappa r^k\right)^d \leq \left(1 + 2\,\mathrm{diam}\, C + \frac{1}{3}\kappa\right)^d r^{kd},$$

which completes the proof. □

The following result is well-known in Fourier Analysis, and we provide a detailed proof for the convenience of the reader. As usual, the support of f, $\mathrm{spt}\, f$, is the closure of $\{x : f(x) \neq 0\}$.

Lemma 4.3 *For any $\kappa > 0$, there exists a non-negative function $h : \mathbb{R}^d \to \mathbb{R}$ such that $h \geq 1$ on $B(0, 1)$ and $\widehat{h}(x) = 0$ for $|x| \geq \kappa$.*

Proof Let ψ be a Schwartz function on $\mathbb{R}^d$ for which $\psi \geq 0$, $\mathrm{spt}\,\psi \subset B(0, 1)$ and $\int \psi = 2$. Let $\eta = |\widehat{\psi}(x)|^2$. Then $\widehat{\eta} = \psi * \widetilde{\psi}$ and therefore, $\mathrm{spt}\,\widehat{\eta} \subset B(0, 1)$, where $\widetilde{\psi}(x) = \psi(-x)$. Hence, both η and $\widehat{\eta}$ are non-negative, and $\eta(0) = \widehat{\psi}(0)^2 = (\int \psi)^2 = 4$. It follows from the continuity that there exists a constant $\kappa_0 < \kappa$ such that $\eta(x) \geq 1$ for any $|x| \leq \kappa_0$. Define $h(x) = \eta(\kappa_0 x)$. Then h is non-negative and $h \geq 1$ on $B(0, 1)$. Since $\mathrm{spt}\,\widehat{\eta} \subset B(0, 1)$ and

$$\widehat{h}(x) = \kappa_0^{-d}\widehat{\eta}(\kappa_0^{-1}x),$$

we have $\mathrm{spt}\,\widehat{h} \subset B(0, \kappa_0)$. Noting that $\kappa_0 < \kappa$, the statement follows. □

We shall first prove the following simplified version of the Main Theorem (iii), which confirms the Conjecture 1.8.

Proposition 4.4 *Assume that the OSC is satisfied. Assume further that $r_1 = \cdots = r_N = r$ and $A_1 = \cdots = A_N = A$. Let μ be the in-homogeneous self-similar measure with the condensation measure ν. Then*

$$\underline{\Delta}_2(\nu) \geq \underline{\Delta}_2(\mu) \geq \min\left\{\frac{u}{2}, \underline{\Delta}_2(\nu)\right\}.$$

Proof Without loss of generality, we may assume that $\underline{\Delta}_2(\nu) > 0$. Fix $\epsilon > 0$. It follows from the definition of $\underline{\Delta}_2(\nu)$ that there exists a constant $c > 0$ such that

$$\frac{1}{\mathcal{L}^d(B(0, R))}\int_{B(0,R)} |\widehat{\nu}(L^k x)|^2\, \mathrm{d}x \leq c(r^k R)^{-2(\underline{\Delta}_2(\nu)-\epsilon)},$$

for all $R > 0$ and $k \in \mathbb{N}$.

Fix $\alpha > 0$. We first prove that there exist constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\alpha)$ such that

$$\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L^k x)\right|^2 \mathrm{d}x \\ \le C_1 r^{ku} r^{2(n-k)(\underline{\Delta}_2(\nu)-\epsilon)} + C_2 r^{(n-k)\alpha}, \tag{4.2}$$

for any integers k and n with $0 \le k \le n$. Define the function $\psi : \mathbb{R}^d \to \mathbb{R}$ by

$$\psi(x) := \begin{cases} \exp\left(\frac{1}{|x|^2-2}+1\right) & \text{if } |x| < 2; \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

It is easy to verify that ψ is a non-negative, infinitely differentiable function on $\mathbb{R}^d$ with the following properties: (i) $\psi(x) \ge 1$ on $B(0,1)$; (ii) spt $\psi \subset B(0,2)$; (iii) $\psi(x) \le 3$ for all $x \in \mathbb{R}^d$. Then we have

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L^k x)\right|^2 \mathrm{d}x \\ &\le \frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\psi(r^n x)\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L^k x)\right|^2 \mathrm{d}x \\ &\le \frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int \psi(r^n x)\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L^k x)\right|^2 \mathrm{d}x \\ &\le \left|\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \mathrm{dist}(S_{\mathbf{j}_1}C,S_{\mathbf{j}_2}C)\le r^k}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\int \psi(r^n x)e^{i\langle x,b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}|\widehat{\nu}(L^k x)|^2\,\mathrm{d}x\right| \\ &\quad + \left|\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \mathrm{dist}(S_{\mathbf{j}_1}C,S_{\mathbf{j}_2}C)> r^k}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\int \psi(r^n x)e^{i\langle x,b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}|\widehat{\nu}(L^k x)|^2\,\mathrm{d}x\right|. \end{aligned} \tag{4.3}$$

By Lemma 4.2 and our choice of ψ, we have

$$\begin{aligned} &\left|\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \mathrm{dist}(S_{\mathbf{j}_1}C,S_{\mathbf{j}_2}C)\le r^k}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\int \psi(r^n x)e^{i\langle x,b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}|\widehat{\nu}(L^k x)|^2\,\mathrm{d}x\right| \\ &\le \sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \mathrm{dist}(S_{\mathbf{j}_1}C,S_{\mathbf{j}_2}C)\le r^k}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\cdot\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,2r^{-n})}|\psi(r^n x)e^{i\langle x,b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}\widehat{\nu}(L^k x)^2|\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)\le r^k}} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \left(\frac{2^d}{\mathcal{L}^d(B(0, c_2 r^{-n}))} \int_{B(0,2r^{-n})} 3|\widehat{\nu}(L^k x)^2| \, \mathrm{d}x \right) \\
&\le \left(\sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)\le r^k}} \frac{p_{\mathbf{j}_1}^2 + p_{\mathbf{j}_2}^2}{2} \right) 2^d \cdot 3c(2r^{k-n})^{-2(\underline{\Delta}_2(\nu)-\epsilon)} \\
&\le C_1 r^{ku} r^{2(n-k)(\underline{\Delta}_2(\nu)-\epsilon)},
\end{aligned}
\tag{4.4}
$$

where $C_1 = 3 \cdot 2^{d-2(\underline{\Delta}_2(\nu)-\epsilon)} N_0 c$, and where the last inequality follows from the fact that

$$
\begin{aligned}
\sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)\le r^k}} p_{\mathbf{j}_1}^2 + p_{\mathbf{j}_2}^2 &= \sum_{|\mathbf{j}_1|=k} \sum_{\substack{|\mathbf{j}_2|=k \\ \operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)\le r^k}} p_{\mathbf{j}_1}^2 + \sum_{|\mathbf{j}_2|=k} \sum_{\substack{|\mathbf{j}_1|=k \\ \operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)\le r^k}} p_{\mathbf{j}_2}^2 \\
&\le N_0 \sum_{|\mathbf{j}_1|=k} p_{\mathbf{j}_1}^2 + N_0 \sum_{|\mathbf{j}_2|=k} p_{\mathbf{j}_2}^2 \\
&= 2N_0 \cdot r^{ku}.
\end{aligned}
$$

Using the definition of Fourier transform and Fubini's theorem, we have

$$
\begin{aligned}
&\int \psi(r^n x) e^{i\langle x, b_{\mathbf{j}_1} - b_{\mathbf{j}_2}\rangle} |\widehat{\nu}(L^k x)|^2 \, \mathrm{d}x \\
&\quad = \int \psi(r^n x) e^{i\langle x, b_{\mathbf{j}_1} - b_{\mathbf{j}_2}\rangle} \left(\int e^{i\langle L^k x, y\rangle} \mathrm{d}\nu(y) \right) \left(\int e^{i\langle L^k x, -z\rangle} \mathrm{d}\nu(z) \right) \mathrm{d}x \\
&\quad = \iiint \psi(r^n x) e^{i\langle x, S_{\mathbf{j}_1} y - S_{\mathbf{j}_2} z\rangle} \, \mathrm{d}x \mathrm{d}\nu(y) \mathrm{d}\nu(z) \\
&\quad = r^{-nd} \iint \widehat{\psi}(r^{-n}(S_{\mathbf{j}_1} y - S_{\mathbf{j}_2} z)) \mathrm{d}\nu(y) \mathrm{d}\nu(z),
\end{aligned}
\tag{4.5}
$$

for all $|\mathbf{j}_1| = |\mathbf{j}_2| = k$. Moreover, if $\operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C) > r^k$, then

$$
|r^{-n}(S_{\mathbf{j}_1} y - S_{\mathbf{j}_2} z)| \ge r^{-n} \cdot \operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C) > r^{k-n},
$$

for any $y, z \in C$. Since ψ is compactly supported infinitely differentiable function, it follows that ψ is a Schwartz function on $\mathbb{R}^d$ and, thus, its Fourier transform $\widehat{\psi}$ is also a Schwartz function. Then there exists a constant $C_\alpha > 0$ such that $|\widehat{\psi}(x)| \le C_\alpha |x|^{-\alpha}$ for all $x \in \mathbb{R}^d$. Hence,

$$
\left| \frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)> r^k}} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \int \psi(r^n x) e^{i\langle x, b_{\mathbf{j}_1} - b_{\mathbf{j}_2}\rangle} |\widehat{\nu}(L^k x)|^2 \, \mathrm{d}x \right|
$$

$$\leq \sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \mathrm{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)>r^k}} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \cdot \frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \left| \int \psi(r^n x) e^{i\langle x, b_{\mathbf{j}_1} - b_{\mathbf{j}_2}\rangle} |\widehat{\nu}(L^k x)|^2 \, \mathrm{d}x \right|$$

$$\leq \sum_{\substack{|\mathbf{j}_1|=|\mathbf{j}_2|=k \\ \mathrm{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)>r^k}} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \cdot \frac{1}{\mathcal{L}^d(B(0, 1))} \iint \left|\widehat{\psi}(r^{-n}(S_{\mathbf{j}_1} y - S_{\mathbf{j}_2} z))\right| \mathrm{d}\nu(y) \mathrm{d}\nu(z)$$

$$\leq \frac{C_\alpha}{\mathcal{L}^d(B(0, 1))} \cdot r^{(n-k)\alpha}. \tag{4.6}$$

Combining this with (4.3) and (4.4), the inequality (4.2) follows.

We then show that there exists a constant $C_3 > 0$ such that

$$\frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \int_{B(0, r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 \mathrm{d}x \leq C_3 r^{ku}, \tag{4.7}$$

for any integers k and n with $0 \leq k \leq n$. Let κ be the constant given by Lemma 4.1, and let $h : \mathbb{R}^d \to \mathbb{R}$ be the function for which the conclusion of Lemma 4.3 holds with κ. Then we have

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \int_{B(0, r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 \mathrm{d}x \\ &\quad \leq \frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \int_{B(0, r^{-n})} h(r^n x) \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 \mathrm{d}x \\ &\quad \leq \frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \int h(r^n x) \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 \mathrm{d}x \\ &\quad \leq \frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \sum_{|\mathbf{j}_1|=|\mathbf{j}_2|=k} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \int h(r^n x) e^{i\langle x, b_{\mathbf{j}_1} - b_{\mathbf{j}_2}\rangle} \mathrm{d}x \\ &\quad = \frac{1}{\mathcal{L}^d(B(0, 1))} \sum_{|\mathbf{j}_1|=|\mathbf{j}_2|=k} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \cdot \widehat{h}(r^{-n}(b_{\mathbf{j}_1} - b_{\mathbf{j}_2})). \end{aligned} \tag{4.8}$$

If $|\mathbf{j}_1| = |\mathbf{j}_2| = k$ with $\mathbf{j}_1 \neq \mathbf{j}_2$, by Lemma 4.1, one has $|r^{-n}(b_{\mathbf{j}_1} - b_{\mathbf{j}_2})| \geq r^{-n}\kappa r^k \geq \kappa$ and therefore, by Lemma 4.3,

$$\widehat{h}(r^{-n}(b_{\mathbf{j}_1} - b_{\mathbf{j}_2})) = 0.$$

Combining this with (4.8), we obtain that

$$\frac{1}{\mathcal{L}^d(B(0, r^{-n}))} \int_{B(0, r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 \mathrm{d}x$$

$$
\begin{aligned}
&\leq \frac{1}{\mathcal{L}^d(B(0,1))} \sum_{|\mathbf{j}_1|=|\mathbf{j}_2|=k} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \cdot \widehat{h}(r^{-n}(b_{\mathbf{j}_1} - b_{\mathbf{j}_2})) \\
&= \left(\frac{\widehat{h}(0)}{\mathcal{L}^d(B(0,1))} \right) \sum_{|\mathbf{j}|=k} p_{\mathbf{j}}^2 \\
&= \left(\frac{\widehat{h}(0)}{\mathcal{L}^d(B(0,1))} \right) \cdot r^{ku}. \qquad (4.9)
\end{aligned}
$$

This completes the proof of (4.7).

Finally, let us estimate $\underline{\Delta}_2(\mu)$. Fix $\rho \in (0,1)$. It follows from (4.1) that

$$
\begin{aligned}
|\widehat{\mu}(x)| &\leq \left| \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L^n x) \right| + p \sum_{k=0}^{n-1} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L^k x) \right| \\
&\leq \sum_{k=0}^{[\rho n]} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L^k x) \right| + \sum_{k=[\rho n]+1}^{n} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|,
\end{aligned}
\qquad (4.10)
$$

where we write $[x]$ for the largest integer less than $x \in \mathbb{R}$. Using Minkowski inequality, we have

$$
\left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} |\widehat{\mu}(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \leq S_1 + S_2, \qquad (4.11)
$$

where

$$
S_1 = \sum_{k=0}^{[\rho n]} \left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L^k x) \right|^2 \mathrm{d}x \right)^{\frac{1}{2}},
$$

and

$$
S_2 = \sum_{k=[\rho n]+1}^{n} \left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.
$$

Putting $\alpha = \frac{\rho u}{1-\rho}$, it follows from (4.2) that

$$
\begin{aligned}
S_1 &\leq \sum_{k=0}^{[\rho n]} \left(C_1 r^{ku} r^{2(n-k)(\underline{\Delta}_2(\nu)-\epsilon)} + C_2 r^{(n-k)\alpha} \right)^{\frac{1}{2}} \\
&\leq \sum_{k=0}^{[\rho n]} \left(C_1^{\frac{1}{2}} r^{k(\frac{u}{2})} r^{(n-k)(\underline{\Delta}_2(\nu)-\epsilon)} + C_2^{\frac{1}{2}} r^{(n-k)\frac{\alpha}{2}} \right)
\end{aligned}
$$

$$\leq (\rho n + 1)\left(C_1^{\frac{1}{2}} r^{n\cdot\min\{\frac{u}{2}, \underline{\Delta}_2(\nu)-\epsilon\}} + C_2^{\frac{1}{2}} r^{(n-\rho n)\frac{\alpha}{2}}\right)$$

$$\leq C_1^{\frac{1}{2}}\left(\frac{1}{r^n}\right)^{-\min\{\frac{u}{2}-\epsilon, \underline{\Delta}_2(\nu)-2\epsilon\}} + C_2^{\frac{1}{2}}\left(\frac{1}{r^n}\right)^{-(\frac{\rho u}{2}-\epsilon)}, \tag{4.12}$$

where, for the last inequality, we used the fact that $\rho n + 1 \leq n + 1 \leq r^{-n\epsilon}$ for n large enough. Similarly, it follows from (4.7) that

$$S_2 \leq \sum_{k=[\rho n]+1}^{n} \left(C_3 r^{ku}\right)^{\frac{1}{2}} \leq \sum_{k=[\rho n]+1}^{n} C_3^{\frac{1}{2}} r^{\rho n(\frac{u}{2})} \leq C_3^{\frac{1}{2}}\left(\frac{1}{r^n}\right)^{-(\frac{\rho u}{2}-\epsilon)}, \tag{4.13}$$

provided that n is large enough. Combining (4.12) and (4.13), together with (4.11), yields

$$\underline{\Delta}_2(\mu) \geq \min\left\{\frac{u}{2} - \epsilon, \underline{\Delta}_2(\nu) - 2\epsilon, \frac{\rho u}{2} - \epsilon\right\}.$$

The statment follows by letting $\epsilon \to 0$ and $\rho \to 1$. □

We can now prove the Main Theorem (iii).

Proof Without loss of generality, we could assume that $\underline{\Delta}_q(\nu) > 0$. Fix $\epsilon > 0$. It follows from the definition of $\underline{\Delta}_q(\nu)$ that there exists a constant $c > 0$ such that

$$\frac{1}{\mathcal{L}^d(B(0,R))}\int_{B(0,R)} |\widehat{\nu}(L^k x)|^q \,\mathrm{d}x \leq c(r^k R)^{-q(\underline{\Delta}_q(\nu)-\epsilon)},$$

for all $R > 0$ and $k \in \mathbb{N}$. Fix $\rho \in (0, 1)$ and recall that

$$|\widehat{\mu}(x)| \leq \sum_{k=0}^{[\rho n]} \left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L^k x)\right| + \sum_{k=[\rho n]+1}^{n} \left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle}\right|.$$

To prove the statement, we need to distinguish two cases.

Case 1 $q > 2$. In this case, using Minkowski inequality, we obtain that

$$\left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})} |\widehat{\mu}(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$

$$\leq \sum_{k=0}^{[\rho n]} \left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})} \left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L^k x)\right|^q \mathrm{d}x\right)^{\frac{1}{q}} \tag{4.14}$$

$$+ \sum_{k=[\rho n]+1}^{n} \left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})} \left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle}\right|^q \mathrm{d}x\right)^{\frac{1}{q}}.$$

Set $\alpha = \frac{\rho u}{1-\rho}$. We follow the proof of (4.2), replacing $\underline{\Delta}_2(\nu)$ therein by $\underline{\Delta}_q(\nu)$, and therefore, we obtain that there exist constants C_1 and C_2 such that

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L^k x)\right|^q \mathrm{d}x \\ &\leq \frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\right|^2 |\widehat{\nu}(L^k x)|^q \mathrm{d}x \\ &\leq C_1 r^{ku} r^{q(n-k)(\underline{\Delta}_q(\nu)-\epsilon)} + C_2 r^{(n-k)\alpha}. \end{aligned} \tag{4.15}$$

Similarly, it follows from (4.7) that there exists a constant C_3 such that

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\right|^q \mathrm{d}x \\ &\leq \frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\right|^2 \mathrm{d}x \leq C_3 r^{ku}. \end{aligned} \tag{4.16}$$

Now we go word by word along the final part of the proof of Proposition 4.4, replacing (4.11), (4.2) and (4.7) therein by (4.14), (4.15) and (4.16) respectively, and we obtain

$$\begin{aligned} &\left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}|\widehat{\mu}(x)|^q \mathrm{d}x\right)^{\frac{1}{q}} \\ &\leq C_1^{\frac{1}{q}}\left(\frac{1}{r^n}\right)^{-\min\{\frac{u}{q}-\epsilon,\underline{\Delta}_q(\nu)-2\epsilon\}} + C_2^{\frac{1}{q}}\left(\frac{1}{r^n}\right)^{-(\frac{\rho u}{q}-\epsilon)} + C_3^{\frac{1}{q}}\left(\frac{1}{r^n}\right)^{-(\frac{\rho u}{q}-\epsilon)}, \end{aligned}$$

provided that n is large enough, and hence, $\underline{\Delta}_q(\mu) \geq \min\{\frac{u}{q}, \underline{\Delta}_q(\nu)\}$.

Case 2 $0 < q < 2$. In this case, by Lemma 2.1, we obtain that

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}|\widehat{\mu}(x)|^q \mathrm{d}x \\ &\leq (n+1)^q \sum_{k=0}^{[\rho n]}\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L^k x)\right|^q \mathrm{d}x \\ &+ (n+1)^q \sum_{k=[\rho n]+1}^{n}\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\right|^q \mathrm{d}x. \end{aligned} \tag{4.17}$$

Set $\alpha = \frac{\rho u}{1-\rho}$. It follows the proof of (4.2) again that there exist constants C_1 and C_2 such that

$$\frac{1}{\mathcal{L}^d(B(0,r^{-n}))}\int_{B(0,r^{-n})}\left|\sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L^k x)\right|^q \mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} \left(\left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 |\widehat{\nu}(L^k x)|^q \right)^{\frac{q}{2}} \cdot |\widehat{\nu}(L^k x)|^{q(1-\frac{q}{2})}\, \mathrm{d}x \\
&\le \left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 |\widehat{\nu}(L^k x)|^q\, \mathrm{d}x \right)^{\frac{q}{2}} \\
&\quad \times \left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} |\widehat{\nu}(L^k x)|^q\, \mathrm{d}x \right)^{\frac{2-q}{2}} \\
&\le \left(C_1 r^{ku} r^{q(n-k)(\underline{\Delta}_q(\nu)-\epsilon)} + C_2 r^{(n-k)\alpha} \right)^{\frac{q}{2}} \cdot c(r^{k-n})^{-q(\underline{\Delta}_q(\nu)-\epsilon)\cdot\frac{2-q}{2}}
\end{aligned} \tag{4.18}
$$

Moreover, it follows from (4.7) that there exists a constant C_3 such that

$$
\begin{aligned}
&\left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^q \mathrm{d}x \right)^{\frac{1}{q}} \\
&\quad \le \left(\frac{1}{\mathcal{L}^d(B(0,r^{-n}))} \int_{B(0,r^{-n})} \left| \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \le C_3 r^{\frac{ku}{2}}.
\end{aligned} \tag{4.19}
$$

Again, we go word by word along the final part of the proof of Proposition 4.4, replacing (4.11), (4.2) and (4.7) therein by (4.17), (4.18) and (4.19) respectively, and we obtain that, if $0 < q < 2$,

$$
\underline{\Delta}_q(\mu) \ge \min\{\frac{u}{2}, \underline{\Delta}_q(\nu)\}.
$$

This completes the proof. □

Remark 4.5 Moreover, in our Main Theorem and Proposition 4.4, the assumptions that all the contractions S_j are equal up to translations can be replaced by a slightly weaker assumption that all the rotations A_j generate a finite group with m elements. Indeed, define

$$
< A_j >= \{B_1, \ldots, B_m\}.
$$

We can always write the iterative formula as a sum of m sums, that is,

$$
\begin{aligned}
\widehat{\mu}(x) &= \sum_{|\mathbf{j}|=n} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L_{\mathbf{j}} x) + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L_{\mathbf{j}} x) \\
&= \sum_{l=1}^{m} \left(\sum_{|\mathbf{j}|=n, A_{\mathbf{j}}=B_l} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(r^n B_l x) + p \sum_{k=0}^{n-1} \sum_{|\mathbf{j}|=k, A_{\mathbf{j}}=B_l} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(r^k B_l x) \right).
\end{aligned}
$$

Then we can repeat the above argument for each of the m smaller sums, which leads to the same result.

5 The Condensation Open Set Condition

In this section, we prove Theorem 1.11. Therefore, we assume that the IFS, $\{S_j\}_{j=1}^N$, together with the condensation set C, satisfies the COSC, i.e. there exists a non-empty, open and bounded set U such that $S_j(U) \subset U$ for all j, $S_j(U) \cap S_k(U) = \emptyset$ for all $j \neq k$, and such that

$$C \subset U \setminus \bigcup S_j(\overline{U}).$$

We would like to emphasize that we will not assume that the equi-contractive condition holds in this section, and thus, all the contracting ratios, $\{r_{\mathbf{j}} : \mathbf{j} \in \mathcal{I}^n\}$, of strings of length n are not comparable as n goes to ∞. However, we can more or less replace the equi-contractive condition and the set $\mathcal{I}^n$ by introducing the concept of δ-*stopping set*.

Given $\delta \in (0, 1)$, we define the δ-*stopping set*, Γ_δ, by

$$\Gamma_\delta = \left\{\mathbf{j} \in \mathcal{I}^* : r_{\mathbf{j}} \leq \delta < r_{\mathbf{j}_-}\right\},$$

where $\mathbf{j}_- = (j_1, \ldots, j_{k-1})$ for $\mathbf{j} = (j_1, \ldots, j_k)$. Then it is easy to see that

$$\sum_{\mathbf{j}\in\Gamma_\delta} p_{\mathbf{j}}^2 r_{\mathbf{j}}^{-u} = 1. \tag{5.1}$$

Furthermore, let Λ_δ denote the set of all prefixes of strings in Γ_δ, i.e. $\Lambda_\delta = \{\mathbf{j} \in \mathcal{I}^* : r_{\mathbf{j}} > \delta\}$. Using this notation, it follows from (1.7) that

$$\mu = \sum_{\mathbf{j}\in\Gamma_\delta} p_{\mathbf{j}} \mu \circ S_{\mathbf{j}}^{-1} + p \sum_{\mathbf{j}\in\Lambda_\delta} p_{\mathbf{j}} \nu \circ S_{\mathbf{j}}^{-1},$$

and thus,

$$|\widehat{\mu}(x)| \leq \left|\sum_{\mathbf{j}\in\Gamma_\delta} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L_{\mathbf{j}} x)\right| + \left|\sum_{\mathbf{j}\in\Lambda_\delta} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L_{\mathbf{j}} x)\right|. \tag{5.2}$$

Observe that the δ-stopping set Γ_δ plays in the general case the similar role the set $\mathcal{I}^n$ plays in the equi-contractive case.

The following result is a simple consequence of the COSC.

Lemma 5.1 *Assume that the IFS, together with the condensation set C, satisfies the COSC. Then there exists a constant $\kappa > 0$ such that, for any $\mathbf{i}, \ \mathbf{j} \in \mathcal{I}^*$ with $\mathbf{i} \neq \mathbf{j}$, one* *has*

$$\mathrm{dist}(S_{\mathbf{i}}C, S_{\mathbf{j}}C) \geq \kappa \cdot \min\{r_{\mathbf{i}}, r_{\mathbf{j}}\}.$$

Proof Without loss of generality, we could assume that $|\mathbf{j}| \geq |\mathbf{i}|$. Let U be the open set given by the COSC. If $\mathbf{i}$ is a prefix of $\mathbf{j}$, then it follows from the COSC that $S_{\mathbf{j}}\overline{U} \cap S_{\mathbf{i}}C = \emptyset$ and $S_{\mathbf{j}}C \subset S_{\mathbf{j}}U$. Hence,

$$\mathrm{dist}(S_{\mathbf{i}}C, S_{\mathbf{j}}C) \geq \mathrm{dist}(S_{\mathbf{j}}C, \partial S_{\mathbf{j}}U) = r_{\mathbf{j}}\,\mathrm{dist}(C, \partial U).$$

If $\mathbf{i}$ is not a prefix of $\mathbf{j}$, then $S_{\mathbf{i}}U \cap S_{\mathbf{j}}U = \emptyset$. Hence, we also obtain that

$$\mathrm{dist}(S_{\mathbf{i}}C, S_{\mathbf{j}}C) \geq \mathrm{dist}(S_{\mathbf{j}}C, \partial S_{\mathbf{j}}U) = r_{\mathbf{j}}\,\mathrm{dist}(C, \partial U).$$

The statement follows by setting $\kappa = \mathrm{dist}(C, \partial U)$. □

Lemma 5.2 *Assume that the IFS, together with the condensation set C, satisfies the COSC. Let U be the open set given by the COSC. Given $\kappa > 0$, there exists a positive integer N_0 such that for any $\delta > 0$ and any $\mathbf{i} \in \Gamma_\delta$, one has*

$$\#\left\{\mathbf{j} \in \Gamma_\delta : \mathrm{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U}) \leq \kappa\delta\right\} \leq N_0.$$

Proof Since U is open and bounded, we could assume that U contains a ball of radius a, and is contained in a ball of radius b. Recall that $r_{\min} = \min_j r_j$. It follows from the definition of δ-stopping set that, for any $\mathbf{j} \in \Gamma_\delta$,

$$r_{\min}\delta < r_{\mathbf{j}} \leq \delta,$$

which implies that each $S_{\mathbf{j}}U, \mathbf{j} \in \Gamma_\delta$, contains a ball of radius $ar_{\min}\delta$, and is contained in a ball of radius $b\delta$. Fix $\mathbf{i} \in \Gamma_\delta$. If $\mathrm{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U}) \leq \kappa\delta$ for some $\kappa > 0$, then all of these $S_{\mathbf{j}}\overline{U}$ are contained in a ball of radius $(4b + \kappa)\delta$. Then, summing the volume of the corresponding interior balls of radius $ar_{\min}\delta$, it follows that

$$\#\left\{\mathbf{j} \in \Gamma_\delta : \mathrm{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U}) \leq \kappa\delta\right\} \cdot (ar_{\min}\delta)^d \leq ((4b + \kappa)\delta)^d,$$

which completes the proof. □

We can now prove Theorem 1.11. Note that its proof is similar to that of Proposition 4.4.

Proof Without loss of generality, we may assume that $\underline{\Delta}_2(\nu) > 0$. Fix $\epsilon > 0$. It follows from the definition of $\underline{\Delta}_2(\nu)$ that there exists a constant $c > 0$ such that

$$\frac{1}{\mathcal{L}^d(B(0, R))}\int_{B(0,R)} |\widehat{\nu}(L_{\mathbf{j}}x)|^2\,\mathrm{d}x \leq c(r_{\mathbf{j}}R)^{-2(\underline{\Delta}_2(\nu)-\epsilon)},$$

for all $R > 0$ and $\mathbf{j} \in \mathcal{I}^*$. Fix $\rho \in (0, 1)$. It follows from (5.2) that, for any $\delta > 0$,

$$|\widehat{\mu}(x)| \leq \left|\sum_{\mathbf{j}\in\Gamma} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L_{\mathbf{j}}x)\right| + \left|\sum_{\mathbf{j}\in\Lambda_1} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L_{\mathbf{j}}x)\right| + \left|\sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L_{\mathbf{j}}x)\right|, \tag{5.3}$$

where $\Gamma = \Gamma_\delta$ is the δ-stopping set, $\Lambda_1 = \{\mathbf{j} \in \mathcal{I}^* : \delta < r_{\mathbf{j}} \leq \delta^\rho\}$ and $\Lambda_2 = \{\mathbf{j} \in \mathcal{I}^* : \delta^\rho < r_{\mathbf{j}}\}$. To prove the statement, it suffices to show that there exist constants $C_1, C_2, C_3 > 0$ such that, for any $\delta > 0$ small enough, we have

$$\left(\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\int_{B(0,\delta^{-1})}\left|\sum_{\mathbf{j}\in\Gamma} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L_{\mathbf{j}}x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C_1 \left(\frac{1}{\delta}\right)^{-\frac{u}{2}}, \tag{5.4}$$

$$\left(\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\int_{B(0,\delta^{-1})}\left|\sum_{\mathbf{j}\in\Lambda_1} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L_{\mathbf{j}}x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C_2 \left(\frac{1}{\delta}\right)^{-\frac{\rho u-\epsilon}{2}}, \tag{5.5}$$

and

$$\left(\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\int_{B(0,\delta^{-1})}\left|\sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\nu}(L_{\mathbf{j}}x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C_3 \left(\frac{1}{\delta}\right)^{-\min\left\{\underline{\Delta}_2(\nu)-2\epsilon,\, \frac{\rho u-\epsilon}{2}\right\}}. \tag{5.6}$$

Indeed, combining these three inequalities with (5.3, we deduce that

$$\underline{\Delta}_2(\mu) \geq \min\left\{\frac{u}{2}, \underline{\Delta}_2(\nu) - 2\epsilon, \frac{\rho u - \epsilon}{2}\right\}.$$

Hence, the desired result follows by letting $\epsilon \to 0$ and $\rho \to 1$.

We shall first prove that

$$\sum_{\mathbf{j}\in\Lambda_1\cup\Lambda_2} p_{\mathbf{j}}^2 r_{\mathbf{j}}^{-u} \leq \delta^{-\epsilon}, \tag{5.7}$$

provided that $\delta > 0$ is small enough. Indeed, for any $\mathbf{j} \in \Lambda_1 \cup \Lambda_2$, we have that $\delta < r_{\mathbf{j}} \leq r_{\max}^{|\mathbf{j}|}$, and thus,

$$|\mathbf{j}| \leq \frac{\log\delta}{\log r_{\max}} \leq \delta^{-\epsilon},$$

provided that $\delta > 0$ is small enough. Hence,

$$\sum_{\mathbf{j}\in\Lambda_1\cup\Lambda_2} p_{\mathbf{j}}^2 r_{\mathbf{j}}^{-u} \le \sum_{k=1}^{[\delta^{-\epsilon}]} \sum_{|\mathbf{j}|=k} p_{\mathbf{j}}^2 r_{\mathbf{j}}^{-u} = \sum_{k=1}^{[\delta^{-\epsilon}]} \left(\sum_{j=1}^{N} p_j^2 r_j^{-u} \right)^k \le \delta^{-\epsilon}.$$

This proves the inequality (5.7).

To prove (5.4), let $\kappa > 0$ be the constant given by Lemma 5.1, and $h : \mathbb{R}^d \to \mathbb{R}$ the function such that the conclusion of Lemma 4.3 holds with κ. Then

$$\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))} \int_{B(0,\delta^{-1})} \left| \sum_{\mathbf{j}\in\Gamma} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L_{\mathbf{j}}x) \right|^2 \mathrm{d}x \\
&\le \frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))} \int_{B(0,\delta^{-1})} h(\delta x) \left| \sum_{\mathbf{j}\in\Gamma} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L_{\mathbf{j}}x) \right|^2 \mathrm{d}x \\
&\le \frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))} \int h(\delta x) \left| \sum_{\mathbf{j}\in\Gamma} p_{\mathbf{j}} e^{i\langle x, b_{\mathbf{j}}\rangle} \widehat{\mu}(L_{\mathbf{j}}x) \right|^2 \mathrm{d}x \\
&\le \frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))} \sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Gamma \\ \mathrm{dist}(S_{\mathbf{j}_1}\overline{U}, S_{\mathbf{j}_2}\overline{U}) \le \kappa\delta}} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \int h(\delta x)\, \mathrm{d}x \\
&\quad + \left| \frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))} \sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Gamma \\ \mathrm{dist}(S_{\mathbf{j}_1}\overline{U}, S_{\mathbf{j}_2}\overline{U}) > \kappa\delta}} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \int h(\delta x) e^{i\langle x, b_{\mathbf{j}_1} - b_{\mathbf{j}_2}\rangle} \widehat{\mu}(L_{\mathbf{j}_1}x) \overline{\widehat{\mu}(L_{\mathbf{j}_2}x)}\, \mathrm{d}x \right|.
\end{aligned} \tag{5.8}$$

It follows from Lemma 5.2 that

$$\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))} \sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Gamma \\ \mathrm{dist}(S_{\mathbf{j}_1}\overline{U}, S_{\mathbf{j}_2}\overline{U}) \le \kappa\delta}} p_{\mathbf{j}_1} p_{\mathbf{j}_2} \int h(\delta x)\, \mathrm{d}x \\
&\le \frac{\widehat{h}(0)}{\mathcal{L}^d(B(0,\delta^{-1}))} \sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Gamma \\ \mathrm{dist}(S_{\mathbf{j}_1}\overline{U}, S_{\mathbf{j}_2}\overline{U}) \le \kappa\delta}} \frac{p_{\mathbf{j}_1}^2 + p_{\mathbf{j}_2}^2}{2} \\
&\le \frac{N_0 \widehat{h}(0)}{\mathcal{L}^d(B(0,\delta^{-1}))} \sum_{\mathbf{j}\in\Gamma} p_{\mathbf{j}}^2 \le \frac{N_0 \widehat{h}(0)}{\mathcal{L}^d(B(0,\delta^{-1}))} \cdot \delta^u,
\end{aligned} \tag{5.9}$$

where the last inequality follows from (5.1). Consider $\mathbf{j}_1, \mathbf{j}_2 \in \Gamma$ with $\mathrm{dist}(S_{\mathbf{j}_1}\overline{U}, S_{\mathbf{j}_2}\overline{U}) > \kappa\delta$. Arguing as in (4.5), using $\mathrm{spt}\,\mu \subset \overline{U}$ and our choice of h, we have

$$\begin{aligned}
&\int h(\delta x) e^{i\langle x, b_{\mathbf{j}_1} - b_{\mathbf{j}_2}\rangle} \widehat{\mu}(L_{\mathbf{j}_1}x) \overline{\widehat{\mu}(L_{\mathbf{j}_2}x)}\, \mathrm{d}x \\
&= \iint \delta^{-d} \cdot \widehat{h}(\delta^{-1}(S_{\mathbf{j}_1}y - S_{\mathbf{j}_2}z)) \mathrm{d}\mu(y) \mathrm{d}\mu(z) = 0.
\end{aligned}$$

Combining this with (5.8) and (5.9), the inequality (5.4) follows.

Let us prove (5.5). An argument similar to the proof of the inequality (5.4) suggests that

$$
\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\int_{B(0,\delta^{-1})}\left|\sum_{\mathbf{j}\in\Lambda_1} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L_{\mathbf{j}}x)\right|^2 \mathrm{d}x\\
&\le \sum_{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_1} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\cdot\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\int h(\delta x)e^{i\langle x,b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}\widehat{\nu}(L_{\mathbf{j}_1}x)\overline{\widehat{\nu}(L_{\mathbf{j}_2}x)}\,\mathrm{d}x\\
&= \sum_{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_1} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\cdot\frac{1}{\mathcal{L}^d(B(0,1))}\iint \widehat{h}(\delta^{-1}(S_{\mathbf{j}_1}y-S_{\mathbf{j}_2}z))\mathrm{d}\nu(y)\mathrm{d}\nu(z).
\end{aligned}
$$

For $\mathbf{j}_1,\mathbf{j}_2\in\Lambda_1$ with $\mathbf{j}_1\neq\mathbf{j}_2$, by Lemma 5.1, we have

$$
\mathrm{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C)\ge\kappa\cdot\min\{r_{\mathbf{j}_1}, r_{\mathbf{j}_2}\}>\kappa\delta,
$$

which implies that $\widehat{h}(\delta^{-1}(S_{\mathbf{j}_1}y-S_{\mathbf{j}_2}z))=0$ for any $y,z\in C$. Recall that spt $\nu=C$. Hence,

$$
\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\int_{B(0,\delta^{-1})}\left|\sum_{\mathbf{j}\in\Lambda_1} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L_{\mathbf{j}}x)\right|^2 \mathrm{d}x\\
&\le \sum_{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_1} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\cdot\frac{1}{\mathcal{L}^d(B(0,1))}\iint \widehat{h}(\delta^{-1}(S_{\mathbf{j}_1}y-S_{\mathbf{j}_2}z))\mathrm{d}\nu(y)\mathrm{d}\nu(z)\\
&=\frac{\widehat{h}(0)}{\mathcal{L}^d(B(0,1))}\sum_{\mathbf{j}\in\Lambda_1}p_{\mathbf{j}}^2\\
&\le\frac{\widehat{h}(0)}{\mathcal{L}^d(B(0,1))}\cdot\delta^{\rho u-\epsilon},
\end{aligned}
$$

where the last inequality follows from the fact that

$$
\sum_{\mathbf{j}\in\Lambda_1}p_{\mathbf{j}}^2\le\left(\sum_{\mathbf{j}\in\Lambda_1}p_{\mathbf{j}}^2 r_{\mathbf{j}}^{-u}\right)\delta^{\rho u}\le\delta^{-\epsilon}\delta^{\rho u}.
$$

This completes the proof of the inequality (5.5).

To prove the inequality (5.6), let ψ be a non-negative, infinitely differentiable function on $\mathbb{R}^d$ with the following properties: (i) $\psi(x)\ge 1$ on $B(0,1)$; (ii) spt $\psi\subset B(0,2)$; (iii) $\psi(x)\le 3$ for all $x\in\mathbb{R}^d$. Arguing as in (4.3), we have

$$
\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\int_{B(0,\delta^{-1})}\left|\sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}} e^{i\langle x,b_{\mathbf{j}}\rangle}\widehat{\nu}(L_{\mathbf{j}}x)\right|^2 \mathrm{d}x
$$

$$\leq \frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}}^2 \int \psi(\delta x)|\widehat{\nu}(L_{\mathbf{j}}x)|^2\,\mathrm{d}x$$
$$+\left|\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_2\\ \mathbf{j}_1\neq\mathbf{j}_2}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\int \psi(\delta x)e^{i\langle x, b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}\widehat{\nu}(L_{\mathbf{j}_1}x)\overline{\widehat{\nu}(L_{\mathbf{j}_2}x)}\,\mathrm{d}x\right|. \tag{5.10}$$

It is easy to see that

$$\begin{aligned}
&\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}}^2 \int \psi(\delta x)|\widehat{\nu}(L_{\mathbf{j}}x)|^2\,\mathrm{d}x\\
&\leq \frac{2^d}{\mathcal{L}^d(B(0,2\delta^{-1}))}\sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}}^2 \int_{B(0,2\delta^{-1})} 3|\widehat{\nu}(L_{\mathbf{j}}x)|^2\,\mathrm{d}x\\
&\leq 3\cdot 2^d \sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}}^2\cdot c(r_{\mathbf{j}}2\delta^{-1})^{-2(\underline{\Delta}_2(\nu)-\epsilon)} \qquad (5.11)\\
&\leq 3\cdot 2^{d-2(\underline{\Delta}_2(\nu)-\epsilon)}c\sum_{\mathbf{j}\in\Lambda_2} p_{\mathbf{j}}^2 r_{\mathbf{j}}^{-u}\cdot r_{\mathbf{j}}^{u-2(\underline{\Delta}_2(\nu)-\epsilon)}\cdot\delta^{2(\underline{\Delta}_2(\nu)-\epsilon)}\\
&\leq 3\cdot 2^{d-2(\underline{\Delta}_2(\nu)-\epsilon)}c\left(\delta^{2\underline{\Delta}_2(\nu)-3\epsilon}+\delta^{\rho u-\epsilon}\right),
\end{aligned}$$

where the last inequality used the inequality (5.7) and the fact that $\delta^\rho < r_{\mathbf{j}} \leq 1$ for any $\mathbf{j}\in\Lambda_2$.

It remains to estimate the second term of the right-hand side of the inequality (5.10). By our choice of ψ, it is well-known that ψ, and hence $\widehat{\psi}$, are both Schwartz functions on $\mathbb{R}^d$. Then there exists a constant $C_\rho > 0$ such that $|\widehat{\psi}(x)| \leq C_\rho |x|^{-\frac{\rho u}{1-\rho}}$ for all $x\in\mathbb{R}^d$. Arguing as (4.5), we obtain that

$$\begin{aligned}
&\left|\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_2\\ \mathbf{j}_1\neq\mathbf{j}_2}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\int \psi(\delta x)e^{i\langle x, b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}\widehat{\nu}(L_{\mathbf{j}_1}x)\overline{\widehat{\nu}(L_{\mathbf{j}_2}x)}\,\mathrm{d}x\right|\\
&\leq \frac{1}{\mathcal{L}^d(B(0,1))}\sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_2\\ \mathbf{j}_1\neq\mathbf{j}_2}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\iint \left|\widehat{\psi}(\delta^{-1}(S_{\mathbf{j}_1}y-S_{\mathbf{j}_2}z))\right|\mathrm{d}\nu(y)\mathrm{d}\nu(z) \qquad (5.12)\\
&\leq \frac{1}{\mathcal{L}^d(B(0,1))}\sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_2\\ \mathbf{j}_1\neq\mathbf{j}_2}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\iint C_\rho\left|\delta^{-1}(S_{\mathbf{j}_1}y-S_{\mathbf{j}_2}z)\right|^{-\frac{\rho u}{1-\rho}}\mathrm{d}\nu(y)\mathrm{d}\nu(z).
\end{aligned}$$

It follows from Lemma 5.1 that for any $\mathbf{j}_1,\mathbf{j}_2\in\Lambda_2$ with $\mathbf{j}_1\neq\mathbf{j}_2$,

$$\operatorname{dist}(S_{\mathbf{j}_1}C, S_{\mathbf{j}_2}C) \geq \kappa\min\{r_{\mathbf{j}_1}, r_{\mathbf{j}_2}\} \geq \kappa\delta^\rho.$$

Hence, we deduce from (5.12) that

$$\left|\frac{1}{\mathcal{L}^d(B(0,\delta^{-1}))}\sum_{\substack{\mathbf{j}_1,\mathbf{j}_2\in\Lambda_2\\ \mathbf{j}_1\neq\mathbf{j}_2}} p_{\mathbf{j}_1}p_{\mathbf{j}_2}\int \psi(\delta x)e^{i\langle x,b_{\mathbf{j}_1}-b_{\mathbf{j}_2}\rangle}\widehat{\nu}(L_{\mathbf{j}_1}x)\overline{\widehat{\nu}(L_{\mathbf{j}_2}x)}\,\mathrm{d}x\right|$$
$$\le \frac{C_\rho \kappa^{-\frac{\rho u}{1-\rho}}}{\mathcal{L}^d(B(0,1))}\cdot\delta^{\rho u}. \tag{5.13}$$

Combining this with inequalities (5.10) and (5.11), the inequality (5.6) follows. □

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