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The Ricci Curvature of Gluing Graph of Two Complete Graphs

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Abstract
We introduce the notion of (k, m)-gluing graph of two complete graphs G_n, G'_n and get an accurate value of the Ricci curvature of each edge on the gluing graph. As an application, we obtain some estimates of the eigenvalues of the normalized graph Laplacian by the Ricci curvature of the (k, m)-gluing graph.

Keywords Ricci curvature · Gluing graph · Coupling · Wasserstein distance

Mathematics Subject Classification Primary: 05C12 · Secondary: 52C99

1 Introduction

The Ricci curvature defined on Riemannian manifold plays an important role in the study of Riemannian geometry, which measures the local amount of non-flatness of the manifold. Using the Wasserstein distance on a metric space (X, d) with a random walk $m = \{m_x\}_{x\in X}$, where each m_x is a probability measure on X, Ollivier introduced the notion of coarse Ricci curvature on metric space in [14]. For two different points $x, y \in X$,

$$\kappa(x, y) := 1 - \frac{W(m_x, m_y)}{d(x, y)},$$

where $W(m_x, m_y)$ is the 1-Wasserstein distance between m_x and m_y. The Ollivier Ricci curvature measures the distance (via the Wasserstein transportation distance) between two small balls centered at two given points. In the paper [10], using the Ricci

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curvature of the lazy random walk, Lin, Lu, and Yau modified notions of Ollivier Ricci curvature to a limit version which is more suitable for graphs. They studied the Ricci curvature over the product space of graphs and random graphs. They also considered a graph with positive Ricci curvature and proved some properties. Since then, the study of Ricci curvature for graphs has been part of a trend in graph theory, and many properties and consequences related to Ollivier Ricci curvature have been obtained on graph by translating notions from Riemannian geometry to graphs [1, 2, 8, 11–13].

Just as in the case of Riemannian geometry, many important results have been obtained on graphs with non-negative Ricci curvature or with Ricci curvature bounded below in [5–7]. Thus, it is meaningful to determine which graph has positive Ricci curvature or to construct a graph with positive Ricci curvature. Part of the difficulty of the study about this problem is lacking of examples where explicit computations are possible. Recently, this problem has been considered by T. Yamada in [17]. It is well known that the complete graph K_n $(n > 2)$ has positive Ricci curvature. In fact, $\kappa(x, y) = n/(n-1)$ for any edge (x, y). In [17], T. Yamada started from two complete graphs K_n and K_n' and connected them by several edges. He called the new graph the m-gluing graph and obtained the following theorem about the least number of edge that result in the m-gluing graph having positive Ricci curvature.

Theorem 1.1 *([17]) For the m-gluing graph $K_n +_m K_n'$ of two complete graphs K_n and K_n' $(n \geq 5)$, we have $\kappa(x, y) > 0$ for any edge $(x, y) \in E$ if and only if*

$$\frac{n^2 - 2n}{n+2} < m \leq n - 1.$$

According the above Theorem, Yamada derived that $m-$gluing graph has positive Ricci curvature if and only if $m \geq n - 2$ for $n = 5, 6$, and $n \geq n - 3$ for $n > 6$. Motivated by the work of T. Yamada, in this paper, we consider the following gluing graph of two complete graphs G_n, G_n', which is called (k, m)-gluing graph. Suppose $G_n = (V(G_n), E(G_n))$ and $G_n' = (V(G_n'), E(G_n'))$ be two complete graphs with vertex sets $V(G_n), V(G_n')$ and edge sets $E(G_n), E(G_n')$, respectively. Suppose $V(G_n) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V(G_n') = \{v_0, v_1, \ldots, v_{n-1}\}$. We define the (k, m)-gluing graph $G = (V(G), E(G))$ as follows:

Definition 1 The (k, m)-*gluing graph* $G_n +_{k,m} G_n'$ of two complete graphs G_n, G_n' is denoted by $G(V, E)$, where (1) the vertex set $V = V(G_n) \cup V(G_n')$, (2) the edge set $E = E(G_n) \cup E(G_n') \cup \{(u_0, v_i), (u_j, v_0) : 0 \leq i \leq k, 0 \leq j \leq m\}$.

We remark that if $k = m$, then the (k, m)-gluing graph is just the m-gluing graph studied by T. Yamada. By the estimates of known results, we calculate the Ricci curvature of each edge based on the definition of the Ricci curvature, we obtain the following theorem regarding the minimal number of k, m satisfying the condition that the (k, m)-gluing graph has positive Ricci curvature.

Main Theorem *Assume that $k = \min\{k, m\}$ and $m = \max\{k, m\}$. Then the (k, m)-gluing graph $G_n +_{k,m} G_n'$ has positive Ricci curvature if and only if one of the following conditions is satisfied:*

(1) $k = 0$, $m = n - 1$ *and* $n > 2$*;*
(2) $k = 1$ *and* $n = 2$*;*
(3) $k \geq n - 2$ *and* $2 < n \leq 6$*;*
(4) $k \geq n - 3$ *and* $n > 6$.

The organization of the paper is as follows: In Sect. 2, we recall some basic facts about Ricci curvature on graphs. In Sect. 3, we calculate the Ricci curvature of the (k, m)-gluing graphs and give the proof of the main result. In the last section, we combine our result and the previous researches, and obtain some estimates of the eigenvalues of the normalized graph Laplacian by the Ricci curvature on the (k, m)-gluing graph.

2 Ricci Curvature

Consider an undirected simple finite graph G, denote $V(G)$ and $E(G)$ be the vertex set and the edge set of G, respectively. For simplicity, we often omit G in $V(G)$ and $E(G)$. For any $u, v \in V$, we write (u, v) as an edge connecting u and v and set the neighborhood of $u \in V$ by

$$\Gamma(u) = \{v \in V : (u, v) \in E\}.$$

Denote the local degree of G at u by d_u, that is, d_u is the cardinality of the set $\Gamma(u)$. We recall that G is a complete graph if there exists an edge in $E(G)$ connecting any two vertices in $V(G)$.

Definition 2 Define m_u a probability measure on V by

$$m_u(v) = \begin{cases} \dfrac{1}{d_u}, & \text{if } v \in \Gamma(u), \\ 0, & \text{otherwise}, \end{cases}$$

where d_u is the degree of u, which is equal to the cardinality of the $\Gamma(u)$. We call $m = \{m_u\}_{u \in V}$ the simple random walk.

In this paper, we use Ollivier's method of defining the Ricci curvature, which is defined by the Wasserstein distance with a random walk $m = \{m_u\}_{u \in V}$; see [14].

Definition 3 The 1-Wasserstein distance between any two probability measure μ and ν on V is given as follows:

$$W(\mu, \nu) = \inf_A \sum_{u, v \in V} A(u, v) d(u, v),$$

where $d(x, y)$ is the graph distance between two vertices x, y and $A : V \times V \to [0, 1]$ runs over all maps satisfying

$$\begin{cases} \sum_{v\in V} A(u, v) = \mu(u), \\ \sum_{u\in V} A(u, v) = \nu(v). \end{cases}$$

Such a map A is called a *coupling* between μ and ν.

Remark For any two probability measure μ and ν on V, there exists a coupling A that attains the 1-Wasserstein distance $W(\mu, \nu)$, and we call it an optimal coupling [9, 15, 16].

A function f on V is said to be 1-*Lipschitz* if $|f(u) - f(v)| \le d(u, v)$ for any $u, v \in V$. Using 1-Lipschitz functions, Kantorovich and Rubinstein gave the equivalent definition of the 1-Wasserstein distance. It provides us a way to calculate the lower bound.

Proposition 2.1 *The* 1-*Wasserstein distance between any two probability measure* μ *and* ν *on* V *is written as follows:*

$$W(\mu, \nu) = \sup_f \sum_{u\in V} f(u)(\mu(u) - \nu(u)),$$

where the supremum is taken over all 1-Lipschitz functions on V.

We use similar definition of Ricci curvature as in [14], which can describe the degree of connection between two distinct vertices.

Definition 4 For two distinct vertices $u, v \in V$, the *Ricci curvature* of u and v is defined by

$$\kappa(u, v) = 1 - \frac{W(m_u, m_v)}{d(u, v)},$$

where $W(m_u, m_v)$ is the 1-Wasserstein distance between m_u and m_v.

We are interested in whether the (k, m)-gluing graph $G(V, E)$ has positive Ricci curvature, which requires $\kappa(u, v) > 0$ for any pair of vertices $(u, v) \in V \times V$. The following proposition implies that it is sufficient to check each edge in E.

Proposition 2.2 *(Ollivier [14]) If* $\kappa(u, v) \ge k$ *for any edge* $(u, v) \in E$ *and for a real number* k*, then* $\kappa(u, v) \ge k$ *for any pair of vertices* $(x, y) \in V \times V$.

Jost and Liu gave the following two theorems which give us a way to estimate the upper and lower bounds of the Ricci curvature.

Theorem 2.3 *(Jost-Liu [8]) On a locally finite graph, for any pair of neighboring vertices u and v, we have*

$$\kappa(u,v) \geq -(1-\frac{1}{d_u}-\frac{1}{d_v}-\frac{\sharp(u,v)}{d_u \wedge d_v})_+ - (1-\frac{1}{d_u}-\frac{1}{d_v}-\frac{\sharp(u,v)}{d_u \vee d_v})_+ + \frac{\sharp(u,v)}{d_u \vee d_v},$$

where $\sharp(u,v)$ is the number of triangles which includes u, v as vertices, and $s_+ := max(s,0)$, $s \vee t := max(s,t)$, and $s \wedge t := min(s,t)$ for real numbers s and t.

Theorem 2.4 *(Jost-Liu [8]) On a locally finite graph, for any pair of neighboring vertices u and v, we have*

$$\kappa(u,v) \leq \frac{\sharp(u,v)}{d_u \vee d_v}.$$

3 Ricci Curvature of (k, m)-Gluing Graph

For simplicity of presentation, we set

$$\Gamma_k(u_0) = \{v_i \in V(G'_n) : 1 \leq i \leq k\} \quad \text{and} \quad \Gamma_m(v_0) = \{u_j \in V(G_n) : 1 \leq j \leq m\},$$

for the (k, m)-gluing graph $G(V, E) = G_n +_{k,m} G'_n$.

We discuss this question by first investigating the trivial case of $n = 2$ and $k = m = 1$.

Proposition 3.1 *For $n = 2$, the gluing graph has positive Ricci curvature if and only if $k = m = 1$.*

Proof We first claim that the gluing graph $G_2 +_{1,1} G'_2$ has positive Ricci curvature. In fact, by the symmetry of the (m, k)-gluing graph, we just need to verify $\kappa(u_0, v_0)$ and $\kappa(u_0, u_1)$. We obtain directly from the Theorem 2.3 and 2.4 that

$$\kappa(u_0, v_0) = \frac{2}{3}, \kappa(u_0, u_1) = \frac{1}{3}.$$

We next prove that the gluing graph $G_2 +_{0,1} G'_2$ does not have positive Ricci curvature. A simple observation gives that $\sharp(v_0, v_1) = 0$, and by Theorem 2.4, we get $\kappa(v_0, v_1) \leq 0$.

Similarly, for the gluing graph $G_2 +_{0,0} G'_2$, we have $\kappa(v_0, v_1) \leq 0$. That complete the proof. □

We only consider the case that $n>2$ in the following discussion. Due to Proposition 2.2 and the symmetry of the gluing graph, it is sufficient to calculate the following Ricci curvatures: $\kappa(u_1, u_2), \kappa(u_1, u_{n-1}), \kappa(u_{n-1}, u_{n-2}), \kappa(u_0, v_0), \kappa(u_0, v_1), \kappa(u_0, u_1)$ and $\kappa(u_0, u_{n-1})$.

From Theorems 2.3 and 2.4, we can immediately obtain

$$\kappa(u_1, u_2) = \begin{cases} \frac{n-2}{n-1}, & \text{if } m = 0, \\ \frac{n-2}{n}, & \text{if } m = 1, \\ \frac{n-1}{n}, & \text{if } m > 1. \end{cases}$$

$$\kappa(u_1, u_{n-1}) = \begin{cases} \frac{n-2}{n-1}, & \text{if } m = 0, \\ \frac{n-2}{n}, & \text{if } 0 < m < n-1, \\ \frac{n-1}{n}, & \text{if } m = n-1. \end{cases}$$

and

$$\kappa(u_{n-1}, u_{n-2}) = \begin{cases} \frac{n-2}{n-1}, & \text{if } m = 0, \\ \frac{n-2}{n-1}, & \text{if } 0 < m < n-2, \\ \frac{n-2}{n}, & \text{if } m = n-2, \\ \frac{n-1}{n}, & \text{if } m = n-1. \end{cases}$$

T.Yamada obtained the following result for the m-gluing graph in [17].

Lemma 3.2 *(Yamada [17]) For vertices u_0 and v_0, we have*

$$\kappa(u_0, v_0) = \begin{cases} \frac{2n-2}{2n-1}, & \textit{if } m = k = n-1, \\ \frac{4m-2n+4}{n+m}, & \textit{if } m = k < n-1. \end{cases}$$

Now, we calculate $\kappa(u_0, v_0)$ for the (k, m)-gluing graph.

Proposition 3.3 *For $u_0 \in V(G_n)$ and $v_0 \in V(G'_n)$, we have*

$$\kappa(u_0, v_0) = \begin{cases} \dfrac{2n-2}{2n-1}, & \textit{if } m = k = n-1, \\ \dfrac{2n-3}{2n-1}, & \textit{if } m = n-1, k = n-2 \textit{ or } k = n-1, m = n-2, \\ \dfrac{(2n+m+k)(m+k-n+2)}{(n+m)(n+k)}, & \textit{if } k, m < n-1, \\ \dfrac{k^2+(3n-1)k+2n-1}{(2n-1)(n+k)}, & \textit{if } m = n-1, k < n-2, \\ \dfrac{m^2+(3n-1)m+2n-1}{(2n-1)(n+m)}, & \textit{if } k = n-1, m < n-2. \end{cases}$$

Proof The case of $m = k$ has been shown in Lemma 3.2; thus, our problem reduces to the case of $m \neq k$. Due to the symmetry, we need consider only the case of $k < m$.

Let us first consider the case of $m < n-1$. Define a map $A_1 : V \times V \to \mathbb{R}$ by

$$A_1(x,y)=\begin{cases}\frac{1}{n+m}, & \text{if } x=y\in\Gamma(u_0)\cap\Gamma(v_0),\\ \frac{1}{(n+m)(n-m-1)}, & \text{if } x\in V(G_n)\setminus\Gamma(v_0),\ y=u_0,\\ \frac{1}{n-k-1}\left(\frac{1}{n+k}-\frac{1}{n+m}\right), & \text{if } x\in\Gamma_m(v_0)\cup\Gamma_k(u_0),\ y\in V(G_n')\setminus\Gamma(u_0),\\ \frac{1}{(n+k)(n-k-1)}, & \text{if } x=v_0,\ y\in V(G_n')\setminus\Gamma(u_0),\\ \frac{1}{n-k-1}[\frac{1}{n+k}-\frac{1}{(n+m)(n-m-1)}], & \text{if } x\in V(G_n)\setminus\Gamma(v_0),\ y\in V(G_n')\setminus\Gamma(u_0),\\ 0, & \text{otherwise}.\end{cases}$$

It is easy to check that A_1 satisfies the condition in Definition 3 of 1-Wasserstein distance, that is, for all $u \in V$, we have

$$\begin{cases}\sum_{v\in V} A_1(u,v)=m_{u_0}(u),\\ \sum_{u\in V} A_1(v,u)=m_{v_0}(u).\end{cases}$$

Hence, we obtain that

$$\begin{aligned}&W(m_{u_0},m_{v_0})\\ &\quad=\inf_A\sum_{u,v\in V}A(u,v)d(u,v)\\ &\quad\le(n-m-1)\frac{1}{(n+m)(n-m-1)}\\ &\qquad+2m(n-k-1)\frac{1}{n-k-1}\left(\frac{1}{n+k}-\frac{1}{n+m}\right)\\ &\qquad+k(n-k-1)\frac{1}{n-k-1}\left(\frac{1}{n+k}-\frac{1}{n+m}\right)+(n-k-1)\frac{1}{(n+k)(n-k-1)}\\ &\qquad+3(n-m-1)(n-k-1)\frac{1}{n-k-1}\left[\frac{1}{n+k}-\frac{1}{(n+m)(n-m-1)}\right]\\ &\quad=\frac{3n^2-4n-m^2-mk-2m-k^2-2k}{(n+m)(n+k)}.\end{aligned}\tag{3.1}$$

To prove the other side of the inequality (3.1), we construct a function on V by

$$f_1(\omega)=\begin{cases}3, & \text{if } \omega\in V(G_n)\setminus\Gamma(v_0),\\ 2, & \text{if } \omega\in\Gamma_m(v_0)\cup\{u_0\},\\ 1, & \text{if } \omega\in\Gamma_k(u_0)\cup\{v_0\},\\ 0, & \text{otherwise}.\end{cases}$$

It is easy to verify that f_1 is a 1-Lipschitz function. According to Proposition 2.1, it follows that

$$\begin{aligned}W(m_{u_0}, m_{v_0}) &= \sup_f \sum_{u\in V} f(u)(m_{u_0}(u) - m_{v_0}(u))\\ &\geq \frac{3(n-m-1)}{n+k} + 2m\left(\frac{1}{n+k} - \frac{1}{n+m}\right)\\ &\quad - \frac{2}{n+m} + k\left(\frac{1}{n+k} - \frac{1}{n+m}\right) + \frac{1}{n+k}\\ &= \frac{3n^2 - 4n - m^2 - mk - 2m - k^2 - 2k}{(n+m)(n+k)}. \end{aligned} \tag{3.2}$$

Inequalities (3.1) and (3.2) lead to

$$W(m_{u_0}, m_{v_0}) = \frac{3n^2 - 4n - m^2 - mk - 2m - k^2 - 2k}{(n+m)(n+k)}.$$

By Definition 4, we derive

$$\kappa(u_0, v_0) = \frac{(2n+m+k)(m+k-n+2)}{(n+m)(n+k)}. \tag{3.3}$$

The next step is considering the case of $m = n-1$ and $k < n-2$. This case can be proved by the similar method as employed in the last case. Construct a coupling between m_{u_0} and m_{v_0} by

$$A_2(x, y) = \begin{cases} \frac{1}{2n-1}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(v_0),\\ \frac{1}{(2n-1)(n-1)}, & \text{if } x \in \Gamma_{n-1}(v_0),\ y = u_0,\\ \frac{1}{n-k-1}\left(\frac{1}{n+k} - \frac{1}{2n-1}\right), & \text{if } x \in \Gamma_k(u_0),\ y \in V(G_n') \setminus \Gamma(u_0),\\ \frac{1}{(n+k)(n-k-1)}, & \text{if } x = v_0,\ y \in V(G_n') \setminus \Gamma(u_0),\\ \frac{1}{n-k-1}[\frac{1}{n+k} - \frac{1}{2n-1} - \frac{1}{(2n-1)(n-1)}], & \text{if } x \in \Gamma_{n-1}(v_0),\ y \in V(G_n') \setminus \Gamma(u_0),\\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}W(m_{u_0}, m_{v_0}) &\leq \sum_{u,v\in V} A_2(u, v)d(u, v)\\ &= \frac{1}{2n-1} + k\left(\frac{1}{n+k} - \frac{1}{2n-1}\right) + \frac{1}{n+k}\\ &\quad + 2(n-1)\left[\frac{1}{n+k} - \frac{1}{2n-1} - \frac{1}{(2n-1)(n-1)}\right]\\ &= \frac{2n^2 - 3n - kn - k^2 + 1}{(n+k)(2n-1)}. \end{aligned}$$

Define a 1-Lipschitz function by

$$f_2(\omega)=\begin{cases}2, & \text{if } \omega\in\Gamma_{n-1}(v_0),\\ 1, & \text{if } \omega\in\Gamma_k(u_0)\cup\{v_0,u_0\},\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, the function constructed above can directly deduce the inequality on the other side, that is,

$$\begin{aligned}W(m_{u_0},m_{v_0}) &\geq \sum_{u\in V} f_2(u)(m_{u_0}(u)-m_{v_0}(u))\\ &=2(n-1)\left(\frac{1}{n+k}-\frac{1}{2n-1}\right)+k\left(\frac{1}{n+k}-\frac{1}{2n-1}\right)\\ &\quad+\frac{1}{n+k}-\frac{1}{2n-1}\\ &=\frac{2n^2-3n-kn-k^2+1}{(n+k)(2n-1)}.\end{aligned}$$

Then

$$W(m_{u_0},m_{v_0})=\frac{2n^2-kn-3n-k^2+1}{(n+k)(2n-1)}.$$

It follows that

$$\kappa(u_0,v_0)=\frac{k^2+(3n-1)k+2n-1}{(2n-1)(n+k)}. \tag{3.4}$$

Finally, we have to show the case of $m=n-1$ and $k=n-2$. Evidently, we have $d_{u_0}=2n-2$, $d_{v_0}=2n-1$ and $\sharp(u_0,v_0)=2n-3$. By the inequality in Theorem 2.3, we obtain that

$$\begin{aligned}\kappa(u_0,v_0) &\geq -(1-\frac{1}{2n-2}-\frac{1}{2n-1}-\frac{2n-3}{2n-2})_+\\ &\quad-(1-\frac{1}{2n-2}-\frac{1}{2n-1}-\frac{2n-3}{2n-1})_+ +\frac{2n-3}{2n-1}\\ &=\frac{2n-3}{2n-1}.\end{aligned}$$

Due to Theorem 2.4, it is straightforward to show that

$$\kappa(u_0,v_0)\leq\frac{2n-3}{2n-1}.$$

Thus, we get

$$\kappa(u_0,v_0)=\frac{2n-3}{2n-1}. \tag{3.5}$$

By formulas (3.3), (3.4), (3.5), Lemma 3.2, and the symmetry of the case of $k \geq m$, we conclude that

$$\kappa(u_0, v_0) = \begin{cases} \frac{2n-2}{2n-1}, & \text{if } m = k = n-1, \\ \frac{2n-3}{2n-1}, & \text{if } m = n-1, k = n-2 \text{ or } k = n-1, m = n-2, \\ \frac{(2n+m+k)(m+k-n+2)}{(n+m)(n+k)}, & \text{if } k, m < n-1, \\ \frac{k^2+(3n-1)k+2n-1}{(2n-1)(n+k)}, & \text{if } m = n-1, k < n-2, \\ \frac{m^2+(3n-1)m+2n-1}{(2n-1)(n+m)}, & \text{if } k = n-1, m < n-2. \end{cases}$$

□

Corollary 3.4 *For $u_0 \in V(G_n)$ and $v_0 \in V(G'_n)$, the Ricci curvature $\kappa(u_0, v_0) > 0$ if and only if $m + k > n - 2$.*

Proposition 3.5 *When $k \geq 1$, for any vertex $v \in \Gamma_k(u_0)$, we have*

$$\kappa(u_0, v) = \begin{cases} \frac{-n^2+kn+2n+2k}{n(n+k)}, & \text{if } m \geq \frac{n^2-k^2}{n} - 2, \\ \frac{-2n^2+mn+kn+4n+k^2+2k}{n(n+k)}, & \text{if } m < \frac{n^2-k^2}{n} - 2. \end{cases}$$

Proof For simplicity, we may take $v = v_1$. The proof of this proposition follows in a similar manner, so we omit the simple calculation process. Below we will divide the proof into three parts by discussing the algebraic relationship among k, m and n.

Part(I): First, we assume that $m > n-3$. Based on the above condition, we consider the case of $k < n-1$. Define a coupling $A_1 : V \times V \to \mathbb{R}$ between m_{u_0} and m_{v_1} by

$$A_1(x, y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(v_1), \\ \frac{1}{n+k}, & \text{if } x \in V(G_n) \setminus \Gamma(v_0), y = u_0, \\ \frac{1}{(n-k-1)(n+k)}, & \text{if } x = v_1, y \in V(G'_n) \setminus \Gamma(u_0), \\ \frac{1}{m}\left(\frac{1}{n} - \frac{n-m-1}{n+k}\right), & \text{if } x \in \Gamma_m(v_0), y = u_0, \\ \frac{1}{m}\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x \in \Gamma_m(v_0), y \in (\{v_0\} \cup \Gamma_k(u_0)) \setminus \{v_1\}, \\ \frac{1}{m}[\frac{1}{n} - \frac{1}{(n-k-1)(n+k)}], & \text{if } x \in \Gamma_m(v_0), y \in V(G'_n) \setminus \Gamma(u_0), \\ 0, & \text{otherwise.} \end{cases}$$

Some tedious manipulation yields that

$$W(m_{u_0}, m_{v_1}) \leq \frac{2(n^2-n-k)}{n(n+k)}.$$

In order to get corresponding lower bound for 1-Wasserstein distance between m_{u_0} and m_{v_1}, we construct

$$f_1(\omega) = \begin{cases} -2, & \text{if } \omega \in V(G'_n) \setminus \{v_0, v_1\}, \\ -1, & \text{if } \omega \in \{u_0, v_0, v_1\}, \\ 0, & \text{otherwise} , \end{cases}$$

which can be verified to be a 1-Lipschitz function. Thus we obtain that

$$W(m_{u_0}, m_{v_1}) = \frac{2(n^2 - n - k)}{n(n+k)}.$$

By Definition 4 of the Ricci curvature, we get

$$\kappa(u_0, v_1) = \frac{-n^2 + kn + 2n + 2k}{n(n+k)}.$$

Now we turn to the case of $k = n - 1$, define a coupling between m_{u_0} and m_{v_1} by

$$A_2(x, y) = \begin{cases} \frac{1}{2n-1}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(v_1), \\ \frac{1}{2n-1}, & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y = u_0, \\ \frac{1}{(n-2)(2n-1)}, & \text{if } x = v_1,\ y \in V(G'_n) \setminus \{v_0, v_1\}, \\ \frac{1}{m}\left(\frac{1}{n} - \frac{n-m-1}{2n-1}\right), & \text{if } x \in \Gamma_m(v_0),\ y = u_0, \\ \frac{1}{m}\left(\frac{1}{n} - \frac{1}{2n-1}\right), & \text{if } x \in \Gamma_m(v_0),\ y = v_0, \\ \frac{1}{m}[\frac{1}{n} - \frac{1}{2n-1} - \frac{1}{(n-2)(2n-1)}], & \text{if } x \in \Gamma_m(v_0),\ y \in V(G'_n) \setminus \{v_0, v_1\}, \\ 0, & \text{otherwise} . \end{cases}$$

We omit the calculation process due to its tediousness, we get the upper bound estimate of the 1-Wasserstein distance as follows:

$$W(m_{u_0}, m_{v_1}) \leq \frac{2(n-1)^2}{n(2n-1)}$$

Next, we define a 1-Lipschitz function on V by

$$f_2(\omega) = \begin{cases} -2, & \text{if } \omega \in V(G'_n) \setminus \{v_0, v_1\}, \\ -1, & \text{if } \omega \in \{u_0, v_0, v_1\}, \\ 0, & \text{if otherwise} . \end{cases}$$

It can easily be checked that

$$W(m_{u_0}, m_{v_1}) \geq \frac{2(n-1)^2}{n(2n-1)}.$$

Then

$$W(m_{u_0}, m_{v_1}) = \frac{2(n-1)^2}{n(2n-1)},$$

which leads to

$$\kappa(u_0, v_1) = \frac{3n-2}{n(2n-1)}.$$

Part(II): Second, we discuss the Ricci curvature when $\frac{n^2-k^2}{n} - 2 \leq m \leq n-3$. In more detail, we assume $k < n-1$. Let

$$A_3(x, y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(v_1), \\ \frac{1}{(n-m-1)n}, & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y = u_0, \\ \frac{1}{m}\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x \in \Gamma_m(v_0),\ y = v_0, \\ \frac{1}{k-1}[\frac{1}{n+k} - \frac{1}{n(n-m-1)}], & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y \in \Gamma_k(u_0) \setminus \{v_1\}, \\ \frac{1}{(n-k-1)(n+k)}, & \text{if } x = v_1,\ y \in V(G'_n) \setminus \Gamma(u_0), \\ \frac{1}{m}[\frac{1}{n} - \frac{1}{(n-k-1)(n+k)}], & \text{if } x \in \Gamma_m(v_0),\ y \in V(G'_n) \setminus \Gamma(u_0), \\ \frac{1}{m}[\frac{1}{n} - \frac{1}{n+k} - \frac{n-m-1}{k-1}\left(\frac{1}{n+k} - \frac{1}{n(n-m-1)}\right)], & \text{if } x \in \Gamma_m(v_0),\ y \in \Gamma_k(u_0) \setminus \{v_1\}, \\ 0, & \text{otherwise}, \end{cases}$$

which is a coupling between m_{u_0} and m_{v_1}. An easy calculation gives

$$W(m_{u_0}, m_{v_1}) \leq \frac{2n^2-2n-2k}{n(n+k)}.$$

As we have stated before, we denote a 1-Lipschitz function f_3 by

$$f_3(\omega) = \begin{cases} -2, & \text{if } \omega \in V(G'_n) \setminus \{v_0, v_1\}, \\ -1, & \text{if } \omega \in \{v_0, v_1, u_0\}, \\ 0, & \text{otherwise}. \end{cases}$$

It derives the corresponding lower bound for the 1-Wasserstein distance as follows:

$$W(m_{u_0}, m_{v_1}) \geq \frac{2n^2-2n-2k}{n(n+k)}.$$

Moreover,

$$W(m_{u_0}, m_{v_1}) = \frac{2n^2-2n-2k}{n(n+k)}.$$

It follows that

$$\kappa(u_0, v_1) = \frac{-n^2+kn+2n+2k}{n(n+k)}.$$

We also need to consider the case where k is equal to $n-1$. We claim that $\kappa(u_0, v_1)$ is positive in such a case. In fact, we define a coupling between m_{u_0} and m_{v_1} by

$$A_4(x, y) = \begin{cases} \frac{1}{2n-1}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(v_1), \\ \frac{1}{(n-m-1)n}, & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y = u_0, \\ \frac{1}{m}\left(\frac{1}{n} - \frac{1}{2n-1}\right), & \text{if } x \in \Gamma_m(v_0),\ y = v_0, \\ \frac{1}{n-2}\left[\frac{1}{2n-1} - \frac{1}{(n-m-1)n}\right], & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y \in \Gamma_{n-1}(u_0) \setminus \{v_1\}, \\ \frac{1}{(n-2)(2n-1)}, & \text{if } x = v_1,\ y \in \Gamma_{n-1}(u_0) \setminus \{v_1\}, \\ \frac{1}{n-2}\left[\frac{1}{2n-1} - \frac{1}{m}\left(\frac{1}{n} - \frac{1}{2n-1}\right)\right], & \text{if } x \in \Gamma_m(v_0),\ y \in \Gamma_{n-1}(u_0) \setminus \{v_1\}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus,

$$\begin{aligned} W(m_{u_0}, m_{v_1}) &\leq \frac{1}{n} + \frac{1}{n} - \frac{1}{2n-1} \\ &\quad + 2(n-m-1)\left[\frac{1}{2n-1} - \frac{1}{n(n-m-1)}\right] \\ &= \frac{2(n-1)^2}{n(2n-1)}. \end{aligned}$$

On the other hand, let

$$f_4(\omega) = \begin{cases} 2, & \text{if } \omega \in V(G_n) \setminus \{u_0\}, \\ 1, & \text{if } \omega \in \{v_0, v_1, u_0\}, \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to verify that f_4 is a 1-Lipschitz function. We, therefore, obtain that

$$W(m_{u_0}, m_{v_1}) = \frac{2(n-1)^2}{n(2n-1)}.$$

Consequently, we obtain that

$$\kappa(u_0, v_1) = \frac{3n-2}{n(2n-1)},$$

which leads to the claim.

Part(III): Finally, it remains the case of $m < \frac{n^2-k^2}{n} - 2$. Let us first define the coupling between m_{u_0} and m_{v_1} by

$$A_5(x,y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(v_1), \\ \frac{1}{(n-m-1)n}, & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y = u_0, \\ \frac{1}{m}\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x \in \Gamma_m(v_0),\ y = v_0, \\ \frac{1}{n-m-1}\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y \in \Gamma_k(u_0) \setminus \{v_1\}, \\ \frac{1}{(n-k-1)(n+k)}, & \text{if } x = v_1,\ y \in V(G'_n) \setminus \Gamma(u_0), \\ \frac{1}{n-k-1}[\frac{1}{n+k} - \frac{1}{m}\left(\frac{1}{n} - \frac{1}{n+k}\right)], & \text{if } x \in \Gamma_m(v_0),\ y \in V(G'_n) \setminus \Gamma(u_0), \\ \frac{1}{n-k-1}[\frac{1}{n+k} - \frac{1}{n(n-m-1)} - \frac{k-1}{n-m-1}\left(\frac{1}{n} - \frac{1}{n+k}\right)], & \text{if } x \in V(G_n) \setminus \Gamma(v_0),\ y \in V(G'_n) \setminus \Gamma(u_0), \\ 0, & \text{otherwise}. \end{cases}$$

Thus,

$$W(m_{u_0}, m_{v_1}) \leq \frac{3n^2 - mn - 4n - k^2 - 2k}{n(n+k)}.$$

We are now in a position to define

$$f_5(\omega) = \begin{cases} 3, & \text{if } \omega \in V(G_n) \setminus \Gamma(v_0), \\ 2, & \text{if } \omega \in \Gamma_m(v_0) \cup \{u_0\}, \\ 1, & \text{if } \omega \in \Gamma_k(u_0) \cup \{v_0\}, \\ 0, & \text{otherwise}, \end{cases}$$

which is not hard to verify that f is a 1-Lipschitz function. It leads to

$$W(m_{u_0}, m_{v_1}) \geq \frac{3n^2 - mn - 4n - k^2 - 2k}{n(n+k)}.$$

Then

$$W(m_{u_0}, m_{v_1}) = \frac{3n^2 - mn - 4n - k^2 - 2k}{n(n+k)}.$$

It concludes that

$$\kappa(u_0, v_1) = \frac{-2n^2 + mn + kn + 4n + k^2 + 2k}{n(n+k)}.$$

Recall the discussion above, we get

$$\kappa(u_0, v_1) = \begin{cases} \dfrac{-n^2 + kn + 2n + 2k}{n(n+k)}, & \text{if } m \geq \dfrac{n^2 - k^2}{n} - 2, \\ \dfrac{-2n^2 + mn + kn + 4n + k^2 + 2k}{n(n+k)}, & \text{if } m < \dfrac{n^2 - k^2}{n} - 2. \end{cases}$$

□

The above proposition tells us, for any vertex $v \in \Gamma_k(u_0)$, $\kappa((u_0, v)) > 0$ if and only if

$$m \geq \frac{n^2 - k^2}{n} - 2 \text{ and } k > \frac{n^2 - 2n}{n + 2}$$

or

$$\frac{2n^2 - kn - 4n - k^2 - 2k}{n} < m < \frac{n^2 - k^2}{n} - 2.$$

That is,

$$m > \frac{2n^2 - kn - 4n - k^2 - 2k}{n} \text{ and } k > \frac{n^2 - 2n}{n + 2}. \tag{3.6}$$

As

$$n - 4 < \frac{n^2 - 2n}{n + 2} < n - 2,$$

then

$$k = n - 3 \text{ and } n > 6 \text{ or } k > n - 3.$$

If $k = n - 3$ and $n > 6$, then we obtain $m \geq 3$ by plugging $k = n - 3$ into the inequality (3.6).

If $k = n - 2$, then $m > 0$ holds for all n.

If $k = n - 1$, then $m > -3 + \frac{1}{n}$ always holds.

In conclusion, we have the following result.

Corollary 3.6 *When $k \geq 1$, for any vertex $v \in \Gamma_k(u_0)$, $\kappa(u_0, v)$ is positive if and only if $k = n - 3$, $m \geq 3$, $n > 6$ or $k > n - 3$.*

Similar to Proposition 3.5 and Corollary 3.6, we obtain the following results.

Proposition 3.7 *When $m \geq 1$, for any vertex $u \in \Gamma_m(v_0)$, we have*

$$\kappa(u, v_0) = \begin{cases} \dfrac{-n^2 + mn + 2n + 2m}{n(n + m)}, & \text{if } k \geq \dfrac{n^2 - m^2}{n} - 2, \\ \dfrac{-2n^2 + kn + mn + 4n + m^2 + 2m}{n(n + m)}, & \text{if } k < \dfrac{n^2 - m^2}{n} - 2. \end{cases}$$

Corollary 3.8 *When $m \geq 1$, for any vertex $u \in \Gamma_m(V_0)$, $\kappa(u, v_0) > 0$ if and only if $m = n - 3$, $k \geq 3$, $n > 6$ or $m > n - 3$.*

Proposition 3.9 (1) *When $m \geq 1$, for any vertex $u \in \Gamma_m(v_0)$, we have*

$$\kappa(u_0, u) = \begin{cases} \dfrac{n - 1}{n + k}, & \text{if } k \leq \dfrac{n}{n - 2}, \\ \dfrac{n^2 - kn + 2k}{n(n + k)}, & \text{if } k > \dfrac{n}{n - 2}, \end{cases}$$

thus $\kappa(u_0, u) > 0$.

(2) *When* $k \geq 1$, *for any vertex* $v \in \Gamma_k(u_0)$, *we have*

$$\kappa(v_0, v) = \begin{cases} \dfrac{n-1}{n+m}, & \text{if } m \leq \dfrac{n}{n-2}, \\ \dfrac{n^2 - mn + 2m}{n(n+m)}, & \text{if } m > \dfrac{n}{n-2}, \end{cases}$$

thus, $\kappa(v_0, v) > 0$.

Proof (1) By symmetry, we just need to calculate $\kappa(u_0, u_1)$. The proofs are almost identical, the major change being the substitution of the maps we construct for the estimate of 1-Wasserstein distance between m_{u_0} and m_{u_1}.

Provided that $k \leq \frac{n}{n-2}$, set

$$A_1(x, y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(u_1), \\ \frac{1}{n} - \frac{1}{n+k}, & \text{if } x = u_1, y \in V(G_n) \setminus \{u_0, u_1\}, \\ \frac{1}{n+k} - (n-2)\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x = u_1, y = u_0, \\ \frac{1}{n+k} - \frac{1}{k}\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x \in \Gamma_k(u_0), y = u_0, \\ \frac{1}{k}\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x \in \Gamma_k(u_0), y = v_0, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$f_1(\omega) = \begin{cases} 1, & \text{if } \omega \in \Gamma_k(u_0) \cup \{u_1\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that A_1 is a coupling between m_{u_0} and m_{u_1}, and f_1 is a 1-Lipschitz function over V. The calculation of this case is not complicated so we omit it here. Thus,

$$W(m_{u_0}, m_{u_1}) = \frac{k+1}{n+k}.$$

Furthermore,

$$\kappa(u_0, u_1) = \frac{n-1}{n+k}.$$

If otherwise, that is, $k > \frac{n}{n-2}$. Define a coupling between m_{u_0} and m_{u_1} by

$$A_2(x, y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(u_1), \\ \frac{1}{(n+k)(n-2)}, & \text{if } x = u_1, y \in V(G_n) \setminus \{u_0, u_1\}, \\ \frac{1}{kn}, & \text{if } x \in \Gamma_k(u_0), y = u_0, \\ \frac{1}{k}\left(\frac{1}{n} - \frac{1}{n+k}\right), & \text{if } x \in \Gamma_k(u_0), y = v_0, \\ \frac{1}{k}\left(\frac{1}{n} - \frac{1}{n+k} - \frac{1}{(n-2)(n+k)}\right), & \text{if } x \in \Gamma_k(u_0), y \in V(G_n) \setminus \{u_0, u_1\}, \\ 0, & \text{otherwise}. \end{cases}$$

and a 1-Lipschitz function by

$$f_2(\omega) = \begin{cases} -2, & \text{if } \omega \in V(G_n) \setminus \{u_0, u_1\}, \\ -1, & \text{if } \omega \in \{u_0, u_1, v_0\}, \\ 0, & \text{otherwise .} \end{cases}$$

We derive that

$$W(m_{u_0}, m_{u_1}) = \frac{2k(n-1)}{n(n+k)}.$$

An easy computation gives rise to

$$\kappa(u_0, u_1) = \frac{n^2 - nk + 2k}{n(n+k)}.$$

In summary, we get the conclusion that

$$\kappa(u_0, u_1) = \begin{cases} \dfrac{n-1}{n+k}, & \text{if } k \leq \dfrac{n}{n-2}, \\ \dfrac{n^2 - kn + 2k}{n(n+k)}, & \text{if } k > \dfrac{n}{n-2}, \end{cases}$$

which conclude that $\kappa(u_0, u_1)$ is always positive. We have completed the proof above.

(2) The proof is similar to (1) thus we omit it. □

Up to now we have discussed all cases of edges connecting u_0 and any vertex adjacent to v_0. If m is less than $n-1$, we point out that the edge connecting u_0 and any vertex in $V(G_n) \setminus \Gamma(v_0)$ is not symmetrical to the edges mentioned above. That will be illustrated in the following proposition.

Proposition 3.10 *If $m < n-1$ and $u \in V(G_n) \setminus \Gamma(v_0)$, then*

$$\kappa(u_0, u) = \begin{cases} \dfrac{n-2}{n+k}, & \text{if } \frac{(n-2)k-1}{k+1} \leq m < n-1,\ k \leq \frac{n}{n-2}, \\ \dfrac{n^2 - kn - 2n + 2k + 2}{(n+k)(n-1)}, & \text{if } \frac{n-1}{k+1} \leq m < n-1, k > \frac{n}{n-2}, \\ \dfrac{n^2 - kn - 3n + km + m + 2k + 3}{(n-1)(n+k)}, & \text{if } m < \min\{\frac{(n-2)k-1}{k+1}, \frac{n-1}{k+1}\}. \end{cases}$$

Moreover, $\kappa(u_0, u)$ is always positive.

Proof For ease of notations, we will restrict ourselves to the proof of $\kappa(u_0, u_{n-1})$. Due to the long process of proof, we discuss the cases of $m = n-2$ and $m < n-2$ in the Part(I) and Part(II), respectively.

Part(I): Consider the case of $m = n-2$.

When $k \leq \frac{n}{n-2}$, define a coupling between m_{u_0} and $m_{u_{n-1}}$ by

$$A_1(x,y)=\begin{cases}\frac{1}{n+k}, & \text{if } x=y\in\Gamma(u_0)\cap\Gamma(u_{n-1}),\\ \frac{1}{n+k}, & \text{if } x\in\Gamma_k(u_0),\ y=u_0,\\ \frac{1}{(n-2)(n+k)}, & \text{if } x=v_0,\ y\in\Gamma_{n-2}(v_0),\\ \frac{1}{n-1}-\frac{k}{n+k}, & \text{if } x=u_{n-1},\ y=u_0,\\ \frac{1}{n-1}-\frac{1}{n+k}-\frac{1}{(n-2)(n+k)}, & \text{if } x=u_{n-1},\ y\in\Gamma_{n-2}(v_0),\\ 0, & \text{otherwise}.\end{cases}$$

It turns out that

$$\begin{aligned}W(m_{u_0},m_{u_{n-1}}) &\leq \frac{k}{n+k}+\frac{1}{n+k}+\frac{1}{n-1}-\frac{k}{n+k}\\ &\quad+(n-2)\left[\frac{1}{n-1}-\frac{1}{n+k}-\frac{1}{(n-2)(n+k)}\right]\\ &=\frac{k+2}{n+k}.\end{aligned}$$

On the other hand, we denote a 1-Lipschitz function by

$$f_1(\omega)=\begin{cases}-1, & \text{if } \omega\in V(G_n)\setminus\{u_0\},\\ 0, & \text{otherwise}.\end{cases}$$

Thus, we get

$$W(m_{u_0},m_{u_{n-1}}) \geq \frac{k+2}{n+k}.$$

Consequently,

$$W(m_{u_0},m_{u_{n-1}}) = \frac{k+2}{n+k}.$$

Furthermore,

$$\kappa(u_0,u_{n-1})=\frac{n-2}{n+k}. \tag{3.7}$$

When $k > \frac{n}{n-2}$, we define a coupling $A_2 : V \times V \to \mathbb{R}$ between m_{u_0} and $m_{u_{n-1}}$ by

$$A_2(x,y)=\begin{cases}\frac{1}{n+k}, & \text{if } x=y\in\Gamma(u_0)\cap\Gamma(u_{n-1}),\\ \frac{1}{k(n-1)}, & \text{if } x\in\Gamma_k(u_0),\ y=u_0,\\ \frac{1}{(n-2)(n+k)}, & \text{if } x\in\{u_{n-1},v_0\},\ y\in\Gamma_{n-2}(v_0),\\ \frac{1}{n-2}[\frac{1}{n+k}-\frac{1}{k(n-1)}], & \text{if } x\in\Gamma_k(u_0),\ y\in\Gamma_{n-2}(v_0),\\ 0, & \text{otherwise}.\end{cases}$$

Then the 1-Wasserstein distance is estimated as follows:

$$\begin{aligned} W(m_{u_0}, m_{u_{n-1}}) &\leq \frac{1}{n-1} + \frac{2}{n+k} + 2k\left[\frac{1}{n+k} - \frac{1}{k(n-1)}\right] \\ &= \frac{2k+2}{n+k} - \frac{1}{n-1}. \end{aligned}$$

On the other hand, we construct a 1-Lipschitz function by

$$f_2(\omega) = \begin{cases} 2, & \text{if } \omega \in \Gamma_k(u_0), \\ 1, & \text{if } \omega \in \{u_0, v_0.u_{n-1}\}, \\ 0, & \text{otherwise.} \end{cases}$$

It can easily be checked that

$$W(m_{u_0}, m_{u_{n-1}}) \geq \frac{2k+2}{n+k} - \frac{1}{n-1}.$$

As what we have hoped, we obtain that

$$W(m_{u_0}, m_{u_{n-1}}) = \frac{2k+2}{n+k} - \frac{1}{n-1}.$$

It comes to the conclusion that

$$\kappa(u_0, u_{n-1}) = \frac{n^2 - kn - 2n + 2k + 2}{(n+k)(n-1)}. \tag{3.8}$$

Part(II): We now proceed the case of $m < n-2$ as in the proof above. In this part, we consider a detail analysis of the relationship among m, k and n.

Firstly, we start our discussion by assumption that $\frac{n-1}{k+1} \leq m < \frac{(n-2)k-1}{k+1}$.

Denote a coupling between m_{u_0} and $m_{u_{n-1}}$ by

$$A_3(x, y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(u_{n-1}), \\ \frac{1}{(n-m-2)(n+k)}, & \text{if } x = u_{n-1}, y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ \frac{1}{k(n-1)}, & \text{if } x \in \Gamma_k(u_0), y = u_0, \\ \frac{1}{m(n+k)}, & \text{if } x = v_0, y \in \Gamma_m(v_0), \\ \frac{1}{k}[\frac{1}{n-1} - \frac{1}{n+k} - \frac{1}{m(n+k)}], & \text{if } x \in \Gamma_k(u_0), y \in \Gamma_m(v_0), \\ \frac{1}{k}[\frac{1}{n-1} - \frac{1}{n+k} - \frac{1}{(n-m-2)(n+k)}], & \text{if } x \in \Gamma_k(u_0), y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ 0, & \text{otherwise.} \end{cases}$$

We get the upper bound for 1-Wasserstein distance as follows directly from Definition 3.

$$\begin{aligned} W(m_{u_0}, m_{u_{n-1}}) &\leq \frac{1}{n+k} + \frac{1}{n-1} + \frac{1}{n+k} + 2m\left[\frac{1}{n-1} - \frac{1}{n+k} - \frac{1}{m(n+k)}\right] \\ &\quad + 2(n-m-2)\left[\frac{1}{n} - \frac{1}{n+k} - \frac{1}{(n-m-2)(n+k)}\right] \\ &= \frac{2kn+n-3k-2}{(n-1)(n+k)}. \end{aligned}$$

In order to get the lower bound estimate, we denote

$$f_3(\omega) = \begin{cases} -2, & \text{if } \omega \in V(G_n) \setminus \{u_0, u_{n-1}\}, \\ -1, & \text{if } \omega \in \{u_0, u_{n-1}, v_0\}, \\ 0, & \text{otherwise}, \end{cases}$$

which is not hard to verify that it is a 1-Lipschitz function. So we derive that

$$W(m_{u_0}, m_{u_{n-1}}) \geq \frac{2kn+n-3k-2}{(n-1)(n+k)}.$$

It follows that

$$W(m_{u_0}, m_{u_{n-1}}) = \frac{2kn+n-3k-2}{(n-1)(n+k)}.$$

Thus, we are led to the conclusion that

$$\kappa(u_0, u_{n-1}) = \frac{n^2-2n-kn+2k+2}{(n+k)(n-1)}. \tag{3.9}$$

Second, we carry on the following work by considering the case of $m < \frac{n-1}{k+1}$ and $k > \frac{n}{n-2}$.

We define a coupling $A_4 : V \times V \to \mathbb{R}$ between m_{u_0} and $m_{u_{n-1}}$ by

$$A_4(x,y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(u_{n-1}), \\ \frac{1}{(n-m-2)(n+k)}, & \text{if } x = u_{n-1}, y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ \frac{1}{k(n-1)}, & \text{if } x \in \Gamma_k(u_0), y = u_0, \\ \frac{1}{n-1} - \frac{1}{n+k}, & \text{if } x = v_0, y \in \Gamma_m(v_0), \\ \frac{1}{n-m-2}[\frac{1}{n+k} - m\left(\frac{1}{n-1} - \frac{1}{n+k}\right)], & \text{if } x = v_0, y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ \frac{1}{n-m-2}[\frac{1}{n+k} - \frac{1}{k(n-1)}], & \text{if } x \in \Gamma_k(u_0), y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ 0, & \text{otherwise}. \end{cases}$$

We obtain the inequality

$$\begin{aligned} W(m_{u_0}, m_{u_{n-1}}) \leq & \frac{1}{n+k} + \frac{1}{n-1} + m\left(\frac{1}{n-1} - \frac{1}{n+k}\right) \\ & + 2[\frac{1}{n+k} - m\left(\frac{1}{n-1} - \frac{1}{n+k}\right)] + 2k\left[\frac{1}{n+k} - \frac{1}{k(n-1)}\right] \\ = & \frac{-(k+1)(m-2n+3)}{(n-1)(n+k)}. \end{aligned}$$

To get the other side, we construct a 1-Lipschitz function $f_4 : V \to \mathbb{R}$ by

$$f_4(\omega) = \begin{cases} 2, & \text{if } \omega \in \Gamma_k(u_0) \cup \{v_0\}, \\ 1, & \text{if } \omega \in \{u_0, u_{n-1}\} \cup \Gamma_m(v_0) \cup (V(G'_n) \setminus \Gamma(u_0)), \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$W(m_{u_0}, m_{u_{n-1}}) = \frac{-(k+1)(m-2n+3)}{(n-1)(n+k)}.$$

A routine computation gives rise to the formula

$$\kappa(u_0, u_{n-1}) = \frac{n^2 - kn - 3n + 2k + 3 + m + mk}{(n+k)(n-1)}. \tag{3.10}$$

Third, we discuss the case of $m < \frac{(n-2)k-1}{k+1}$ and $k \leq \frac{n}{n-2}$. Let A_5 be a coupling between m_{u_0} and $m_{u_{n-1}}$ as follows,

$$A_5(x, y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(u_{n-1}), \\ \frac{1}{(n-m-2)(n+k)}, & \text{if } x = u_{n-1},\ y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ \frac{1}{n+k}, & \text{if } x \in \Gamma_k(u_0),\ y = u_0, \\ \frac{1}{n-1} - \frac{k}{n+k}, & \text{if } x = v_0,\ y = u_0, \\ \frac{1}{n-1} - \frac{1}{n+k}, & \text{if } x = v_0,\ y \in \Gamma_m(v_0), \\ \frac{1}{n-1} - \frac{1}{n+k} - \frac{1}{(n-m-2)(n+k)}, & \text{if } x = v_0,\ y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to get

$$W(m_{u_0}, m_{u_{n-1}}) \leq \frac{2kn + 2n - km - m - 3k - 3}{(n-1)(n+k)}.$$

On the other hand, denote f_5 a 1-Lipschitz function of V by

$$f_5(\omega) = \begin{cases} -2, & \text{if } \omega \in V(G'_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ -1, & \text{if } \omega \in \Gamma_m(v_0) \cup \{u_0, u_{n-1}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$W(m_{u_0}, m_{u_{n-1}}) \geq \frac{2kn + 2n - km - m - 3k - 3}{(n-1)(n+k)}.$$

Hence we derive that

$$W(m_{u_0}, m_{u_{n-1}}) = \frac{2kn + 2n - km - m - 3k - 3}{(n-1)(n+k)}.$$

Furthermore,

$$\kappa(u_0, u_{n-1}) = \frac{n^2 - kn - 3n + km + m + 2k + 3}{(n-1)(n+k)}. \tag{3.11}$$

Next, we proceed the proof with the case of $\frac{(n-2)k-1}{k+1} \leq m < n-2$ and $k \leq \frac{n}{n-2}$. Define a coupling between m_{u_0} and $m_{u_{n-1}}$ by

$$A_6(x, y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(u_{n-1}) \\ \frac{1}{n+k}, & \text{if } x \in \Gamma_k(u_0),\ y = u_0, \\ \frac{k+1}{(n-1)(n+k)}, & \text{if } x = u_{n-1},\ y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ \frac{2k+n-kn}{(n-1)(n+k)}, & \text{if } x = v_0,\ y = u_0, \\ \frac{kn-2k-1}{m(n-1)(n+k)}, & \text{if } x = v_0,\ y \in \Gamma_m(v_0), \\ \frac{km+m-kn+2k+1}{m(n-1)(n+k)}, & \text{if } x = u_{n-1},\ y \in \Gamma_m(v_0), \\ 0, & \text{otherwise.} \end{cases}$$

Through a series of simple calculation and simplification, we get

$$W(m_{u_0}, m_{u_{n-1}}) \leq \frac{k+2}{n+k}.$$

Moreover, we define a 1-Lipschitz function

$$f_6(\omega) = \begin{cases} 1, & \text{if } \omega \in \Gamma_k(u_0) \cup \{v_0, u_{n-1}\}, \\ 0, & \text{otherwise.} \end{cases}$$

Due to Proposition 2.1, we obtain

$$W(m_{u_0}, m_{u_{n-1}}) \geq \frac{k+2}{n+k}.$$

Thus,

$$W(m_{u_0}, m_{u_{n-1}}) = \frac{k+2}{n+k}.$$

The Ricci curvature of edge connecting u_0 and u_{n-1} follows by Definition 4, that is,

$$\kappa(u_0, u_{n-1}) = 1 - W(m_{u_0}, m_{u_{n-1}}) = \frac{n-2}{n+k}. \tag{3.12}$$

Finally, it remains to show the case when $\frac{(n-2)k-1}{k+1} \leq m < n-2$ and $k > \frac{n}{n-2}$. We are almost ready to invoke the proof steps above. Let

$$A_7(x,y) = \begin{cases} \frac{1}{n+k}, & \text{if } x = y \in \Gamma(u_0) \cap \Gamma(u_{n-1}), \\ \frac{1}{n-1} - \frac{1}{n+k}, & \text{if } x = u_{n-1}, y \in V(G_n) \setminus (\Gamma(v_0) \cup \{u_{n-1}\}), \\ \frac{1}{k(n-1)}, & \text{if } x \in \Gamma_k(u_0), y = u_0, \\ \frac{1}{m}[\frac{1}{n+k} - (n-m-2) & \\ (\frac{1}{n-1} - \frac{1}{n+k})], & \text{if } x = u_{n-1}, y \in \Gamma_m(v_0), \\ \frac{1}{m(n+k)}, & \text{if } x = v_0, y \in \Gamma_m(v_0), \\ \frac{1}{m}[\frac{1}{n+k} - \frac{1}{k(n-1)}], & \text{if } x \in \Gamma_k(u_0), y \in \Gamma_m(v_0), \\ 0, & \text{otherwise}, \end{cases}$$

which is a coupling between m_{u_0} and $m_{u_{n-1}}$. We derive that

$$\begin{aligned} W(m_{u_0}, m_{u_{n-1}}) \leq & (n-m-2)\left(\frac{1}{n-1} - \frac{1}{n+k}\right) + \frac{1}{n-1} \\ & + \left[\frac{1}{n+k} - (n-m-2)\left(\frac{1}{n-1} - \frac{1}{n+k}\right)\right] \\ & + \frac{1}{n+k} + 2k\left[\frac{1}{n+k} - \frac{1}{k(n-1)}\right] \\ = & \frac{2kn+n-3k-2}{(n-1)(n+k)}. \end{aligned}$$

Next we define a 1-Lipschitz function

$$f_7(\omega) = \begin{cases} 2, & \text{if } \omega \in \Gamma_k(u_0), \\ 1, & \text{if } \omega \in \{u_0, u_{n-1}\} \cup (V(G'_n) \setminus \Gamma_k(u_0)), \\ 0, & \text{otherwise}. \end{cases}$$

It is not difficult to yield that

$$W(m_{u_0}, m_{u_{n-1}}) \geq \frac{2kn+n-3k-2}{(n-1)(n+k)}.$$

Then

$$W(m_{u_0}, m_{u_{n-1}}) = \frac{2kn+n-3k-2}{(n-1)(n+k)}.$$

Therefore, we get the result that

$$\kappa(u_0, u_{n-1}) = \frac{n^2 - kn - 2n + 2k + 2}{(n+k)(n-1)}. \tag{3.13}$$

In view of the formulas (3.7), (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), it follows that

$$\kappa(u_0, u_{n-1}) = \begin{cases} \dfrac{n-2}{n+k}, & \text{if } \frac{(n-2)k-1}{k+1} \le m < n-1,\ k \le \frac{n}{n-2}, \\ \dfrac{n^2 - kn - 2n + 2k + 2}{(n+k)(n-1)}, & \text{if } \frac{n-1}{k+1} \le m < n-1, k > \frac{n}{n-2}, \\ \dfrac{n^2 - kn - 3n + km + m + 2k + 3}{(n-1)(n+k)}, & \text{if } m < \min\{\frac{(n-2)k-1}{k+1}, \frac{n-1}{k+1}\}. \end{cases}$$

At the end of the proof, we consider when $\kappa(u_0, u_{n-1})$ is positive.

① When $\frac{(n-2)k-1}{k+1} \le m < n-1$ and $k \le \frac{n}{n-2}$, it is easy to find that $\kappa(u_0, u_{n-1})$ is always positive.

② When $\frac{n-1}{k+1} \le m < n-1$ and $k > \frac{n}{n-2}$, Since

$$k < n < \frac{n^2 - 2n + 2}{n-2},$$

thus, $\kappa(u_0, u_{n-1}) > 0$ in this case.

③ When $m < \min\{\frac{(n-2)k-1}{k+1}, \frac{n-1}{k+1}\}$. We claim that $k \le n-3$.

If $k = n-1$, then

$$m < \min\{\frac{n^2 - 3n + 1}{n}, \frac{n-1}{n}\} \le \frac{n-1}{n} < 1,$$

which leads to a contradiction.

If $k = n-2$, then

$$m < \min\{\frac{n^2 - 4n + 3}{n-1}, 1\} \le 1,$$

which also leads to a contradiction. That illustrates the claim.

As $(n-3-k)n + 2k + 3 + m + mk > 0$ holds when $k \le n-3$. Then $\kappa(u_0, u_{n-1})$ is also always positive in the third case.

Therefore, we find that Ricci curvature between u_0 and u_{n-1} is always positive for all cases we classify above although the formula may be different. The proof is completed. □

Similar to Proposition 3.10, we have

Proposition 3.11 *If $k < n-1$, then $\kappa(v_0, v)$ is always positive, for any vertex $v \in V(G'_n) \setminus \Gamma(u_0)$, where*

$$\kappa(v_0, v) = \begin{cases} \dfrac{n-2}{n+m}, & \text{if } \frac{(n-2)m-1}{m+1} \le k < n-1,\ m \le \frac{n}{n-2}, \\ \dfrac{n^2-mn-2n+2m+2}{(n+m)(n-1)}, & \text{if } \frac{n-1}{m+1} \le k < n-1, m > \frac{n}{n-2}, \\ \dfrac{n^2-mn-3n+km+k+2m+3}{(n-1)(n+m)}, & \text{if } k < \min\{\frac{(n-2)m-1}{m+1}, \frac{n-1}{m+1}\}. \end{cases}$$

Moreover, $\kappa(v_0, v)$ is always positive.

Remark If $k = 0$ and $m = n-1$ ($n > 2$), then it follows from Proposition 3.11 that $\kappa(v_0, v_1) = \frac{1}{(n-1)(2n-1)}$, which is the smallest Ricci curvature of the $(0, n-1)-$gluing graph.

Proof of Main theorem From the above Propositions 3.1,3.9, 3.10, 3.11, and Corollaries 3.4,3.6, 3.8, we can obtain our main theorem. □

4 Application

In this section, using the lower bound of the Ricci curvature, we derive some estimates of the eigenvalues of the normalized graph Laplacian.

We firstly introduce some notations. Let $G(V, E)$ be a graph with n vertices. Let A be the adjacent matrix of $G(V, E)$ and D be the diagonal matrix of degrees. The normalized Laplacian is matrix $L = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$. The eigenvalues of L are called Laplacian Eigenvalues of G, which are listed as

$$0 = \lambda_0 \le \lambda_1 \le \cdots \le \lambda_{n-1}.$$

In [10], Lin-Lu-Yau gave an estimate of the eigenvalues of the normalized graph Laplacian by the Ricci curvature as follows:

Theorem 4.1 *(Lin-Lu-Yau [10], Theorem 4.2) Suppose G is a finite graph and λ_1 is the first nonzero Laplacian eigenvalue of G. If for any edge (x, y), $\kappa(x, y) \ge \kappa > 0$, then $\lambda_1 \ge \kappa$.*

Second, for the (k, m)-gluing graph, we obtain the maximum lower bound of the Ricci curvature of each edge for every (k, m)-gluing graph with positive curvature.

Theorem 4.2 *Assume that $G_n +_{k,m} G'_n$ is a (k, m)-gluing graph with positive Ricci curvature. Let $\underline{\kappa} = \min_{(x,y)\in E} \kappa(x, y)$ and $k_0 = \min\{k, m\}$. Then*

$$\kappa = \begin{cases} \frac{1}{(n-1)(2n-1)}, & \text{if } k = 0, m = n-1, n > 2,\\ \frac{1}{3}, & \text{if } k = m = 1, n = 2,\\ \frac{2}{5}, & \text{if } k = m = 2, n = 3,\\ \frac{3n-2}{n(2n-1)}, & \text{if } k_0 = n-1, n > 3,\\ \frac{n}{(n-1)(2n-1)}, & \text{if } k_0 = n-2, k \neq m \text{ and } n = 5, 6,\\ \frac{n-2}{n(n-1)}, & \text{if } k_0 = n-2, k \neq m, n \neq 5, 6 \text{ or } k = m = n-2,\\ \frac{n-6}{n(2n-3)}, & \text{if } k_0 = n-3, n > 6. \end{cases}$$

Proof We only give a detailed illustration of the case of $k_0 = n-3$ and $n > 6$, because the other cases can be similarly obtained by substituting values into those formulas we obtained in the Sect. 3. By Propositions and Corollaries in the Sect. 3, we have

$$\begin{aligned} \kappa(u_0, v_0) &= \frac{2(n-4)}{2n-3},\\ \kappa(u_1, u_2) = \kappa(v_1, v_2) &= \frac{n-1}{n},\\ \kappa(u_0, u_1) = \kappa(v_0, v_1) &= \frac{5n-6}{n(2n-3)},\\ \kappa(u_0, v_1) = \kappa(v_0, u_1) &= \frac{n-6}{n(2n-3)}\\ \kappa(u_1, u_{n-1}) = \kappa(v_1, v_{n-1}) &= \frac{n-2}{n},\\ \kappa(u_{n-2}, u_{n-1}) = \kappa(v_{n-2}, v_{n-1}) &= \frac{n-2}{n-1},\\ \kappa(u_0, u_{n-1}) = \kappa(v_0, v_{n-1}) &= \frac{3n-4}{(n-1)(2n-3)}. \end{aligned}$$

Comparing these equalities, and we obtain the minimum Ricci curvature $\underline{\kappa} = \frac{n-6}{n(2n-3)}$ when $k_0 = n-3$ and $n > 6$. □

Due to the above theorem, we obtain following corollary.

Corollary 4.3 *Let $G(V, E) = G_n +_{k,m} G'_n$ be a gluing graph with positive Ricci curvature and λ_1 be the first nonzero Laplacian eigenvalue of $G(V, E)$. Then $\lambda_1 \geq \underline{\kappa}$, where*

$$\kappa = \begin{cases} \frac{1}{(n-1)(2n-1)}, & if\, k = 0, m = n-1, n > 2, \\ \frac{1}{3}, & if\, k = m = 1, n = 2; \\ \frac{2}{5}, & if\, k = m = 2, n = 3; \\ \frac{3n-2}{n(2n-1)}, & if\, k_0 = n-1, n > 3; \\ \frac{n}{(n-1)(2n-1)}, & if\, k_0 = n-2, k \neq m\ and\ n = 5, 6; \\ \frac{n-2}{n(n-1)}, & if\, k_0 = n-2, k \neq m, n \neq 5, 6\ or\ k = m = n-2; \\ \frac{n-6}{n(2n-3)}, & if\, k_0 = n-3, n > 6. \end{cases}$$

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Declarations

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