Research Article

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On the L^q spectra of in-homogeneous self-similar measures

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Abstract: The in-homogeneous self-similar measure μ is defined by the relation

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} + p\nu,$$

where $(p_1, \dots, p_N, p)$ is a probability vector, each $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, N$, is a contraction similarity, and ν is a compactly supported Borel probability measure on $\mathbb{R}^d$. In this paper, we study the L^q-spectra of in-homogeneous self-similar measures. We obtain non-trivial lower and upper bounds for the L^q-spectra of an arbitrary in-homogeneous self-similar measure. Moreover, if the IFS satisfies some separation conditions, the bounds for the L^q-spectra can be improved.

Keywords: In-homogeneous self-similar measure, L^q-spectra, separation conditions

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1 Introduction

Throughout this paper, we always assume that $\mathbf{I} = \{S_1, \dots, S_N\}$ is an *iterated function system* (IFS) of contracting similarities on $\mathbb{R}^d$. That is, $S_j x = r_j A_j x + a_j$, where $0 < r_j < 1$, A_j is an orthogonal matrix and $a_j \in \mathbb{R}^d$, for each $1 \le j \le N$. It is a fundamental result in fractal geometry that there exists a unique, non-empty compact set $K_\emptyset \subset \mathbb{R}^d$ such that (see [16])

$$K_\emptyset = \bigcup_{i=1}^{N} S_i K_\emptyset.$$

We call $K_\emptyset$ the *self-similar set* generated by $\mathbf{I}$. In order to understand the fractal structures of self-similar sets, one studies the so-called self-similar measures. More precisely, given a probability vector $\mathbf{p} = (p_1, \dots, p_N)$, i.e. $p_i > 0$ for each i and $\sum_i p_i = 1$, there exists a unique Borel probability measure μ_0 supported on $K_\emptyset$ such that

$$\mu_0 = \sum_{i=1}^{N} p_i \mu_0 \circ S_i^{-1}.$$

We say that the measure μ_0 is the *self-similar measure* generated by $(\mathbf{I}, \mathbf{p})$ in this paper. Self-similar sets and measures play an important role in the study of fractal geometry, and we refer the reader to [8, 9, 16] and the references therein for the detailed properties of self-similar sets and measures. Observe that the self-similar

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measure μ_0 can be viewed as the unique solution of the following homogeneous equation:

$$\mu - \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} = 0.$$

This viewpoint suggests to us the following natural generalization of self-similar measures.

Definition 1.1. Let $\mathbf{I} = \{S_i\}_{i=1}^N$ be an IFS of similarities, let $\mathbf{p} = (p_1, \dots, p_N, p)$ be a probability vector, and let ν be a Borel probability measure on $\mathbb{R}^d$ with compact support. A Borel probability measure μ satisfying the equation

$$\mu - \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} = p\nu \tag{1.1}$$

is called the *in-homogeneous self-similar measure* generated by $(\mathbf{I}, \mathbf{p}, \nu)$.

In-homogeneous self-similar sets and measures were first introduced and studied by Barnsley and Demko in [4], where they considered some examples of in-homogeneous self-similar measures. In [1–3, 13, 24], in-homogeneous self-similar measures are also called *orbital measures* and the in-homogeneous term ν is called the *condensation measure*. The existence and uniqueness of such measures is well known; see [18]. Furthermore, it has been shown that the support of the in-homogeneous self-similar measure μ is equal to the unique non-empty compact set K_C, called the *in-homogeneous self-similar set*, such that

$$K_C = \bigcup_{i=1}^{N} S_i K_C \cup C,$$

where C is the compact support of the measure ν. In [3], Barnsley defined the orbital set by the union of the condensation set C, with all images of C under compositions of maps in $\mathbf{I}$. It is well known that the in-homogeneous self-similar set K_C is the closure of the orbital set, and it turns out that the orbital set plays an important role in the structure of the in-homogeneous self-similar set.

In recent years, multifractal theory has aroused widespread concern among theoretical physicists and mathematicians. Rényi introduced the Rényi entropies in 1960 in [20–22]. In [14], Hentschel and Procaccia defined the generalized Rényi dimensions and used integrals in an attempt to characterize the class of mean value functions which induce additive entropy functions. The popularity of Rényi dimensions is basically the relation between Rényi dimensions and the multifractal spectra. It was proved by Horbacz, Myjak and Szarek [15] that the multifractal spectrum and the Rényi dimension can be derived from each other. Moreover, the L^q-spectrum is equal to the Rényi dimension with a constant multiple difference when q is finite, so it makes sense for us to study the L^q-spectrum of the in-homogeneous self-similar measure. Now, we recall the notion of L^q-spectrum of the Borel probability measure μ on $\mathbb{R}^d$ for $q \in \mathbb{R}$.

Definition 1.2. For a Borel probability measure μ on $\mathbb{R}^d$ and $q \in \mathbb{R}$, the *lower L^q-spectrum* $\underline{\tau}_\mu(q)$ and the *upper L^q-spectrum* $\overline{\tau}_\mu(q)$ of μ are defined as follows:

$$\underline{\tau}_\mu(q) = \liminf_{r \to 0} \frac{\log \int_{\operatorname{spt}\mu} \mu(B(x,r))^{q-1}\, d\mu(x)}{-\log r}, \tag{1.2}$$

$$\overline{\tau}_\mu(q) = \limsup_{r \to 0} \frac{\log \int_{\operatorname{spt}\mu} \mu(B(x,r))^{q-1}\, d\mu(x)}{-\log r}, \tag{1.3}$$

where $\operatorname{spt}\mu$ denotes the support of μ and $B(x, r)$ denotes the open ball of radius r centered at x.

During the past twenty years, many papers focused on the L^q-spectra of the homogeneous self-similar measures (see, e.g., [5–7, 10–12]). In particular, for an IFS $\mathbf{I} = \{S_i\}_{i=1}^N$ of similarities with contraction ratios $r_i \in (0, 1)$ for $i = 1, \dots, N$ and probability vector $\mathbf{p} = (r_1^s, \dots, r_N^s)$, Shmerkin proved that for any $q > 0$ the L^q-spectrum of the homogeneous self-similar measure μ is affine in [23], that is,

$$\tau_\mu(q) = s(1 - q).$$

However, the results are limited in the in-homogeneous case. An important result on the L^q-spectrum of the in-homogeneous self-similar measures is the following theorem obtained by Olsen and Snigireva in [19].

Theorem 1.3 ([19, Theorem 2.1]). *Assume that the sets* $(S_1K_C, \dots, S_NK_C, C)$ *are pairwise disjoint.*

(i) *For all* $q \in \mathbb{R}$, *we have*

$$\overline{\tau}_\mu(q) \le \max\{\overline{\tau}_\nu(q), \beta(q)\}.$$

(ii) *For all* $q \in \mathbb{R}$, *we have*

$$\min\{\underline{\tau}_\nu(q), \beta(q)\} \le \underline{\tau}_\mu(q).$$

(iii) *For all* $q \ge 1$, *we have*

$$\max\{\underline{\tau}_\nu(q), \beta(q)\} \le \underline{\tau}_\mu(q),$$
$$\max\{\overline{\tau}_\nu(q), \beta(q)\} \le \overline{\tau}_\mu(q).$$

Moreover, in [19], Olsen and Snigireva introduced the *in-homogeneous open set condition* (IOSC) by assuming that there exists a non-empty and bounded open set U such that the following conditions are satisfied:

(C1) $S_iU \subseteq U$ for all i.

(C2) $S_iU \cap C^\circ = \emptyset$ for all i.

(C3) $S_iU \cap S_jU = \emptyset$ for all $i \ne j$.

Here A° denotes the interior of a set $A \subset \mathbb{R}^d$. Then they posed the following questions.

Question 1.4 ([19]). Are the results above true if the IOSC is satisfied?

Motivated by the above question, we study the L^q-spectra of in-homogeneous self-similar measures in this paper. Our main results are Theorems 2.7–2.12. Sections 3 and 4 are devoted to presenting the proofs.

2 Preliminaries and main results

2.1 Basic definitions and notations

Let $\mathbf{I} = \{S_1, \dots, S_N\}$ be an iterated function system of contracting similarities on $\mathbb{R}^d$. Suppose $S_ix = r_iA_ix + a_i$, where $r_i \in (0,1)$, $a_i \in \mathbb{R}^d$ and A_i is an orthogonal matrix for each $1 \le i \le N$. Let $\mathbf{p} = (p_1, \dots, p_N, p)$ be a probability vector with $p > 0$ and let ν be a condensation measure supported on a compact set C. We denote by μ the in-homogeneous self-similar measure generated by $(\mathbf{I}, \mathbf{p}, \nu)$, and by K_C the corresponding in-homogeneous self-similar set. Let $\Sigma = \{1, \dots, N\}$ be the set of alphabets. Denote the set of all finite strings with entries in Σ by

$$\Sigma^* = \{\mathbf{i} = i_1 \cdots i_n : n \in \mathbb{N},\ i_k = 1, \dots, N\},$$

and the set of all strings with length n by

$$\Sigma^n = \{\mathbf{i} = i_1 \cdots i_n : i_k = 1, \dots, N\}.$$

In particular, write $\Sigma^0 = \{\omega\}$, where ω is the empty word and the map S_ω is taken to be the identity. For a finite string $\mathbf{i} = i_1 \dots i_n$, denote the length of $\mathbf{i}$ by $|\mathbf{i}|$, i.e. $|\mathbf{i}| = n$, the restriction of $\mathbf{i}$ to its first entry by $\mathbf{i}|_1 = i_1$, and the restriction of $\mathbf{i}$ to its first $n-1$ entries by $\mathbf{i}_- = (i_1, \dots, i_{n-1})$. We write $S_\mathbf{i} = S_{i_1} \circ \cdots \circ S_{i_n}$. Then $S_\mathbf{i}$ is a contraction similarity on $\mathbb{R}^d$ and has the form

$$S_\mathbf{i}x = r_\mathbf{i}A_\mathbf{i}x + a_\mathbf{i},$$

where $r_\mathbf{i} = r_{i_1} \cdots r_{i_n} \in (0,1)$, $A_\mathbf{i} = A_{i_1} \cdots A_{i_n}$ is an orthogonal matrix and $a_\mathbf{i}$ is a vector in $\mathbb{R}^d$. Similarly, we define $p_\mathbf{i} = p_{i_1} \cdots p_{i_n}$.

For any $n \in \mathbb{N}$, it follows easily from (1.1) that

$$\mu = \sum_{|\mathbf{i}|=n} p_\mathbf{i}\mu \circ S_\mathbf{i}^{-1} + p \sum_{|\mathbf{i}|<n} p_\mathbf{i}\nu \circ S_\mathbf{i}^{-1}. \tag{2.1}$$

It is obvious to see that

$$p \sum_{|\mathbf{i}|<n} p_\mathbf{i}\nu \circ S_\mathbf{i}^{-1} \le \mu \le (1-p)^n + p \sum_{|\mathbf{i}|<n} p_\mathbf{i}\nu \circ S_\mathbf{i}^{-1}.$$

Letting n tend to infinity, we have

$$\mu = p \sum_{\mathbf{i} \in \Sigma^*} p_{\mathbf{i}} \nu \circ S_{\mathbf{i}}^{-1}. \tag{2.2}$$

We recall that an IFS $\mathbf{I} = \{S_i\}_{i=1}^N$ satisfies the *open set condition* (OSC) if there exists a non-empty open set U such that $\bigcup_i S_i U \subseteq U$ and $S_i U \cap S_j U = \emptyset$ for $i \neq j$. The OSC is fundamental in the investigation of the homogeneous case. In order to study the L^q-spectra of in-homogeneous self-similar measures, we need to adapt the OSC to the in-homogeneous case as follows.

Definition 2.1. An IFS $\mathbf{I} = \{S_i\}_{i=1}^N$, together with a condensation set C, satisfies the *condensation open set condition* (COSC) if $\mathbf{I}$ satisfies the OSC and the open set U can be chosen such that $C \subseteq U \setminus (\bigcup_i S_i \overline{U})$.

It is of interest to study the L^q-spectra of the in-homogeneous self-similar measure satisfying the COSC. Moreover, we consider a more general situation by assuming that there exists a non-empty and bounded open set U such that the following conditions are satisfied:
(S1) $C \subseteq \overline{U}$.
(S2) $S_i U \subseteq U$ for all i.
(S3) $S_i U \cap S_j U = \emptyset$ for all $i \neq j$.
(S4) $\nu(\partial U) = \nu(S_i \overline{U}) = 0$ for all i.
Here ∂U is the boundary of U, i.e. $\partial U = \overline{U} \setminus U$.

Remark 2.2. The in-homogeneous self-similar set under the COSC satisfies conditions (S1)–(S4).

Similar to the similarity dimension, we denote by $\beta(q)$ the *q-th dimension* of μ, defined by the solution of the following equation:

$$\sum_i p_i^q r_i^{\beta(q)} = 1. \tag{2.3}$$

It is clear that the function $\beta(q)$ is well defined, that is, for every $q \in \mathbb{R}$ we can find a unique $\beta(q) \in \mathbb{R}$ such that (2.3) holds. Moreover, it is easy to prove that $\beta(q)$ is a convex function, and it is strictly decreasing with q. In particular, we note that $\beta(1) < 0$, implying that $\beta(q) < 0$ for all $q \geq 1$.

Definition 2.3. The *Assouad dimension* of a measure ν is defined by

$$\dim_A \nu = \inf\Big\{s \geq 0 : \text{there exists } C > 0 \text{ such that } \frac{\nu(B(x,R))}{\nu(B(x,r))} \leq C\Big(\frac{R}{r}\Big)^s \text{ for all } 0 < r < R < \operatorname{diam}(\operatorname{spt} \nu) \text{ and } x \in \operatorname{spt} \nu\Big\} \tag{2.4}$$

and, provided $\operatorname{diam}(\operatorname{spt} \nu) > 0$, the *lower dimension* of ν is defined by

$$\dim_L \nu = \sup\Big\{s \geq 0 : \text{there exists } C > 0 \text{ such that } \frac{\nu(B(x,R))}{\nu(B(x,r))} \geq C\Big(\frac{R}{r}\Big)^s \text{ for all } 0 < r < R < \operatorname{diam}(\operatorname{spt} \nu) \text{ and } x \in \operatorname{spt} \nu\Big\}, \tag{2.5}$$

and otherwise it is 0. We adopt the convention that $\inf \emptyset = +\infty$.

Remark 2.4. The restriction $R < \operatorname{diam}(\operatorname{spt} \nu)$ is not required in the definition of $\dim_A \nu$.

The Assouad and lower dimensions of a measure were introduced formally in Käenmäki, Lehrbäck and Vuorinen [17]. When they were first introduced, the Assouad and lower dimensions of measures were referred to as the upper and lower regularity dimensions. Theses dimensions describe the optimal global control on the relative measure of concentric balls.

Definition 2.5. Let ν be a probability measure on $\mathbb{R}^d$. We define the *lower ∞-th Rényi dimension* of ν by

$$\underline{D}_\nu(\infty) = \liminf_{r \to 0} \frac{\log \sup_{x \in \operatorname{spt} \nu} \nu(B(x,r))}{\log r}.$$

Fix a positive integer n. We set

$$M_{\mathbf{i}}^n = \begin{cases} S_{\mathbf{i}}K_C, & |\mathbf{i}| = n, \\ S_{\mathbf{i}}C, & 0 < |\mathbf{i}| < n, \\ C, & \mathbf{i} = \omega, \end{cases}$$

for any $\mathbf{i} \in \bigcup_{k=0}^n \Sigma^k$. Furthermore, we set

$$\mathcal{A}_n = \Big\{\Delta \subseteq \bigcup_{k=0}^n \Sigma^k : \bigcap_{\mathbf{i}\in\Delta} M_{\mathbf{i}}^n \neq \emptyset\Big\}.$$

It is clear that $\{\mathbf{i}\} \in \mathcal{A}_n$ for all $\mathbf{i} \in \bigcup_{k=0}^n \Sigma^k$, and thus $\mathcal{A}_n$ is non-empty. For any $\Delta \in \mathcal{A}_n$, we define a function $\Phi_\Delta : \mathbb{R} \to \mathbb{R}$ by

$$\Phi_\Delta(s) = \sum_{\substack{\mathbf{i}\in\Delta \\ |\mathbf{i}|=n}} p_{\mathbf{i}} r_{\mathbf{i}}^s + p \sum_{\substack{\mathbf{i}\in\Delta \\ |\mathbf{i}|<n}} p_{\mathbf{i}} r_{\mathbf{i}}^s.$$

Since the function $\Phi_\Delta(s)$ is continuous and strictly decreasing with

$$\lim_{s\to-\infty} \Phi_\Delta(s) = +\infty \quad \text{and} \quad \lim_{s\to+\infty} \Phi_\Delta(s) = 0,$$

there exists a unique $s(\Delta) \in \mathbb{R}$ such that

$$1 = \Phi_\Delta(s(\Delta)) = \sum_{\substack{\mathbf{i}\in\Delta \\ |\mathbf{i}|=n}} p_{\mathbf{i}} r_{\mathbf{i}}^{s(\Delta)} + p \sum_{\substack{\mathbf{i}\in\Delta \\ |\mathbf{i}|<n}} p_{\mathbf{i}} r_{\mathbf{i}}^{s(\Delta)}.$$

Let

$$s_n = \sup_{\Delta\in\mathcal{A}_n} s(\Delta).$$

Remark 2.6. s_n is monotonically increasing with respect to n.

2.2 IFS without any separation conditions

First, we significantly generalize the result in [19]. Our results bound the upper and lower L^q-spectra of the in-homogeneous self-similar measures without any separation conditions.

Theorem 2.7. *Let μ be the in-homogeneous self-similar measure generated by $(\mathbf{I}, \mathbf{p}, \nu)$.*
(i) *For all $q \geq 1$, we have*

$$\overline{\tau}_\mu(q) \geq \max\{\beta(q),\ \overline{\tau}_\nu(q)\},$$
$$\underline{\tau}_\mu(q) \geq \max\{\beta(q),\ \underline{\tau}_\nu(q)\}.$$

(ii) *For all $q \leq 1$, we have*

$$\overline{\tau}_\mu(q) \leq \max\{\beta(q),\ \overline{\tau}_\nu(q)\},$$
$$\underline{\tau}_\mu(q) \leq \max\{\beta(q),\ \underline{\tau}_\nu(q),\ \underline{\tau}_\nu(q) + \beta(q) - (1-q)\dim_L \nu\}.$$

We next provide a non-trivial upper bound for $q \geq 1$ and a lower bound for $q \leq 1$.

Theorem 2.8. *Suppose μ is the in-homogeneous self-similar measure generated by $(\mathbf{I}, \mathbf{p}, \nu)$. Let n be the smallest integer satisfying*

$$\bigcap_{|\mathbf{i}|\leq n} M_{\mathbf{i}}^n = \emptyset.$$

(i) *For all $q \geq 1$, we have*

$$\underline{\tau}_\mu(q) \leq \overline{\tau}_\mu(q) \leq \max\{(q-1)s_n,\ (1-q)\underline{D}_\nu(\infty)\}.$$

(ii) *For all $q \leq 1$, we have*

$$\overline{\tau}_\mu(q) \geq \underline{\tau}_\mu(q) \geq \max\{(q-1)s_n,\ (1-q)\underline{D}_\nu(\infty)\}.$$

2.3 IFS with separation conditions

In this subsection, we consider the L^q spectra of the in-homogeneous self-similar measure generated by IFS with some separation conditions.

Theorem 2.9. *Let μ be the in-homogeneous self-similar measure generated by* $(\mathbf{I}, \mathbf{p}, \nu)$. *Assume that*

$$\bigcup_{i=1}^{N} S_i K_C \cap C = \emptyset.$$

Then, for all $q \in \mathbb{R}$,

$$\underline{\tau}_\mu(q) \geq \underline{\tau}_\nu(q), \quad \overline{\tau}_\mu(q) \geq \overline{\tau}_\nu(q).$$

Theorem 2.10. *Assume that* (S1)–(S4) *are satisfied. Then, for all $q \leq 1$,*

$$\overline{\tau}_\mu(q) \geq \underline{\tau}_\mu(q) \geq \beta(q).$$

From Theorems 2.7, 2.9 and 2.10, it is easy to obtain the following theorem.

Theorem 2.11. *Assume that the COSC is satisfied. Then, for all $q \leq 1$,*

$$\overline{\tau}_\mu(q) = \max\{\overline{\tau}_\nu(q),\ \beta(q)\},$$
$$\max\{\underline{\tau}_\nu(q),\ \beta(q)\} \leq \underline{\tau}_\mu(q) \leq \max\{\beta(q),\ \underline{\tau}_\nu(q),\ \underline{\tau}_\nu(q) + \beta(q) - (1-q)\dim_L \nu\}.$$

Finally, Our task now is to estimate the upper and lower L^q-spectra of the in-homogeneous self-similar measure for all $q \geq 1$.

Theorem 2.12. *Assume that the COSC is satisfied. Then, for any $q \geq 1$,*

$$\overline{\tau}_\mu(q) = \max\{\overline{\tau}_\nu(q),\ \beta(q)\},$$
$$\max\{\underline{\tau}_\nu(q),\ \beta(q)\} \leq \underline{\tau}_\mu(q) \leq \max\{\beta(q),\ \underline{\tau}_\nu(q),\ \underline{\tau}_\nu(q) + \beta(q) - (1-q)\dim_A \nu\}.$$

3 IFS without any separation conditions

In this section, we study the L^q-spectra of the in-homogeneous self-similar measures generated by an IFS without any separation conditions. The main goal is to prove Theorems 2.7 and 2.8. First, we will establish several lemmas for the proof of our main results. We denote the diameter of a bounded subset A of $\mathbb{R}^d$ by diam A. For any $r \in (0, 1]$, we define a $\frac{r}{\operatorname{diam} K_C}$-stopping by

$$\Gamma_1(r) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} \leq \frac{r}{\operatorname{diam} K_C} < r_{\mathbf{i}_-}\Big\}, \tag{3.1}$$

and set

$$\Gamma_2(r) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} > \frac{r}{\operatorname{diam} K_C}\Big\}. \tag{3.2}$$

Moreover, we write

$$\Gamma_3(r) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} > \frac{r}{\operatorname{diam} C}\Big\}. \tag{3.3}$$

Obviously, $\Gamma_3(r)$ is a subset of $\Gamma_2(r)$. According to the iterative formula (1.1), we have the following result.

Lemma 3.1. *For any $r \in (0, \operatorname{diam} K_C)$, we have*

$$\mu = \sum_{\mathbf{i} \in \Gamma_1(r)} p_{\mathbf{i}} \mu \circ S_{\mathbf{i}}^{-1} + p \sum_{\mathbf{i} \in \Gamma_2(r)} p_{\mathbf{i}} \nu \circ S_{\mathbf{i}}^{-1}.$$

Denote the minimum and the maximum of $\{r_i\}_i$ by $r_{\min}$ and $r_{\max}$, respectively. For convenience, in this paper, we write

$$C_{\min} = \min\Big\{(\operatorname{diam} K_C)^{\beta(q)},\ (\operatorname{diam} C)^{\beta(q)},\ \Big(\frac{\operatorname{diam} K_C}{r_{\min}}\Big)^{\beta(q)}\Big\}$$

and

$$C_{\max} = \max\Big\{(\operatorname{diam} K_C)^{\beta(q)},\ (\operatorname{diam} C)^{\beta(q)},\ \Big(\frac{\operatorname{diam} K_C}{r_{\min}}\Big)^{\beta(q)}\Big\}.$$

Lemma 3.2. *For any $r \in (0, \operatorname{diam} K_C)$ and $q \in \mathbb{R}$, we have*

$$C_{\min} r^{-\beta(q)} \le \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q \le C_{\max} r^{-\beta(q)}.$$

Proof. Fix $r \in (0, \operatorname{diam} K_C)$ and $q \in \mathbb{R}$. According to the definition of $\Gamma_1(r)$ in (3.1), it follows that for any $\mathbf{i} \in \Gamma_1(r)$,

$$r_{\mathbf{i}} \le \frac{r}{\operatorname{diam} K_C} < r_{\mathbf{i}_-} \le \frac{r_{\mathbf{i}}}{r_{\min}}.$$

Clearly,

$$\frac{r \cdot r_{\min}}{\operatorname{diam} K_C} < r_{\mathbf{i}} \le \frac{r}{\operatorname{diam} K_C}.$$

By repeated application of equation (2.3), we obtain

$$\sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} = 1.$$

Thus,

$$\begin{aligned}\sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q &= \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} r_{\mathbf{i}}^{-\beta(q)} \le \max\Big\{\Big(\frac{r\cdot r_{\min}}{\operatorname{diam} K_C}\Big)^{-\beta(q)},\ \Big(\frac{r}{\operatorname{diam} K_C}\Big)^{-\beta(q)}\Big\}\cdot\Big(\sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)}\Big)\\ &\le C_{\max} r^{-\beta(q)}.\end{aligned}$$

Similarly, we can deduce the inequality on the other side. □

Lemma 3.3. *Given an $\epsilon > 0$, there exists $r_0 \in (0,1)$ such that*

$$\sum_{\mathbf{i}\in\Gamma_3(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} \le \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} < r^{-\epsilon}$$

for any $0 < r < r_0$.

Proof. The first inequality is trivial since $\Gamma_3(r)$ is a subset of $\Gamma_2(r)$. We will show the second inequality in the following. Given an $\epsilon > 0$, it is easy to see that there exists $r_0 \in (0,1)$ such that

$$\max\Big\{1,\ \frac{\log \frac{r}{\operatorname{diam} K_C}}{\log r_{\max}}\Big\} < \frac{1}{2r^\epsilon}$$

for any $0 < r < r_0$. For any $r \in (0, r_0)$ and $\mathbf{i} \in \Gamma_2(r)$, by the definition of $\Gamma_2(r)$ in (3.2), we have

$$r_{\max}^{|\mathbf{i}|} \ge r_{\mathbf{i}} > \frac{r}{\operatorname{diam} K_C},$$

that is,

$$|\mathbf{i}| < \frac{\log \frac{r}{\operatorname{diam} K_C}}{\log r_{\max}} < \frac{1}{2r^\epsilon}.$$

Hence, we conclude that

$$\begin{aligned}\sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} &\le \sum_{|\mathbf{i}|<\frac{1}{2r^\epsilon}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)}\\ &= \sum_{k<\frac{1}{2r^\epsilon}} \sum_{|\mathbf{i}|=k} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)}\\ &= \sum_{k<\frac{1}{2r^\epsilon}} \Big(\sum_i p_i^q r_i^{\beta(q)}\Big)^k\\ &\le \frac{1}{2r^\epsilon} + 1\\ &< \frac{1}{r^\epsilon}.\end{aligned}$$

□

Recall that the lower L^q spectrum and the upper L^q-spectrum of a measure ν are respectively defined by

$$\underline{\tau}_\nu(q) = \liminf_{r\to 0} \frac{\log \int_{\operatorname{spt}\nu} \nu(B(x,r))^{q-1}\,\mathrm{d}\nu(x)}{-\log r}$$

and

$$\overline{\tau}_\nu(q) = \limsup_{r\to 0} \frac{\log \int_{\operatorname{spt}\nu} \nu(B(x,r))^{q-1}\,\mathrm{d}\nu(x)}{-\log r}.$$

For simplicity of presentation, we write

$$I_\nu(q,r) = \int_{\operatorname{spt}\nu} \nu(B(x,r))^{q-1}\,\mathrm{d}\nu(x)$$

for any $q \in \mathbb{R}$ and $r > 0$.

Lemma 3.4. *Let ν be a probability measure supported on a compact set C.*
(i) *For any $q \geq 1$, we have*

$$(1-q)\dim_A \nu \leq \underline{\tau}_\nu(q) \leq \overline{\tau}_\nu(q) \leq (1-q)\dim_L \nu.$$

(ii) *For any $q \leq 1$, we have*

$$(1-q)\dim_L \nu \leq \underline{\tau}_\nu(q) \leq \overline{\tau}_\nu(q) \leq (1-q)\dim_A \nu.$$

Proof. We only prove (a). The proof of (b) is similar and is omitted. Fix $q \geq 1$. We begin by proving the first inequality. Let $t > \dim_A \nu$. By the definition of $\dim_A \nu$ in (2.4), there exists $c_1 > 0$ such that for any $r \in (0, \operatorname{diam} C)$ and $x \in C$,

$$\nu(B(x,r)) \geq c_1 r^t.$$

Therefore,

$$I_\nu(q,r) = \int_C \nu(B(x,r))^{q-1}\,\mathrm{d}\nu(x) \geq c_1^{q-1} r^{t(q-1)}.$$

It follows that $\underline{\tau}_\nu(q) \geq t(1-q)$. Letting $t \to \dim_A \nu$ yields the desired inequality.

The middle inequality is trivial, so it remains to prove the final inequality. Let $s < \dim_L \nu$. By definition of $\dim_L \nu$ in (2.5), there exists $c_2 > 0$ such that for any $r \in (0, \operatorname{diam} C)$ and $x \in C$,

$$\nu(B(x,r)) \leq c_2 r^s.$$

Therefore, we deduce that

$$I_\nu(q,r) = \int_C \nu(B(x,r))^{q-1}\,\mathrm{d}\nu(x) \leq c_2^{q-1} r^{s(q-1)}.$$

It is easy to see that $\overline{\tau}_\nu(q) \leq s(1-q)$. Letting $s \to \dim_L \nu$ gives the result. □

Proof of Theorem 2.7. Consider $0 < r < \operatorname{diam} K_C$. It follows from Lemma 3.1 that for any $q \in \mathbb{R}$,

$$\begin{aligned} I_\mu(q,r) &= \int_{K_C} \mu(B(x,r))^{q-1}\,\mathrm{d}\mu(x) \\ &= \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}} \int_{S_{\mathbf{i}}(K_C)} \mu(B(x,r))^{q-1}\,\mathrm{d}\mu\circ S_{\mathbf{i}}^{-1}x + p \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}} \int_{S_{\mathbf{i}}(C)} \mu(B(x,r))^{q-1}\,\mathrm{d}\nu\circ S_{\mathbf{i}}^{-1}x. \end{aligned} \tag{3.4}$$

By the same token, for any $x \in K_C$ and $r > 0$, we have

$$\begin{aligned} \mu(B(x,r)) &= \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}\mu\circ S_{\mathbf{i}}^{-1}(B(x,r)) + p\sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}\nu\circ S_{\mathbf{i}}^{-1}(B(x,r)) \\ &= \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}\mu\Big(B\Big(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\Big)\Big) + p\sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}\nu\Big(B\Big(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\Big)\Big). \end{aligned}$$

This implies that

$$\mu(B(x,r)) \geq \begin{cases} p_{\mathbf{i}}\mu\Big(B\Big(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\Big)\Big) & \text{for } x \in S_{\mathbf{i}}K_C,\ \mathbf{i}\in\Gamma_1(r), \\ pp_{\mathbf{i}}\nu\Big(B\Big(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\Big)\Big) & \text{for } x \in S_{\mathbf{i}}C,\ \mathbf{i}\in\Gamma_2(r). \end{cases} \tag{3.5}$$

Consider $q \geq 1$. If we plug (3.5) back into (3.4), then we derive

$$\begin{aligned} I_\mu(q,r) &\geq \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q \int_{S_{\mathbf{i}}(K_C)} \mu\Big(B\Big(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\Big)\Big)^{q-1} \mathrm{d}\mu \circ S_{\mathbf{i}}^{-1}x + p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q \int_{S_{\mathbf{i}}(C)} \nu\Big(B\Big(S_{\mathbf{i}}^{-1}x, \frac{r}{r_{\mathbf{i}}}\Big)\Big)^{q-1} \mathrm{d}\nu \circ S_{\mathbf{i}}^{-1}x \\ &= \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q \int_{K_C} \mu\Big(B\Big(x, \frac{r}{r_{\mathbf{i}}}\Big)\Big)^{q-1} \mathrm{d}\mu(x) + p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q \int_{C} \nu\Big(B\Big(x, \frac{r}{r_{\mathbf{i}}}\Big)\Big)^{q-1} \mathrm{d}\nu(x) \\ &= \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q I_\mu\Big(q, \frac{r}{r_{\mathbf{i}}}\Big) + p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r}{r_{\mathbf{i}}}\Big). \end{aligned}$$

It is clear that $\frac{r}{r_{\mathbf{i}}} \geq \operatorname{diam} K_C$ for any $\mathbf{i} \in \Gamma_1(r)$. Then, for any $x \in K_C$, we have

$$K_C \subseteq B\Big(x, \frac{r}{r_{\mathbf{i}}}\Big),$$

which infers that $\mu(B(x, \frac{r}{r_{\mathbf{i}}})) = 1$. Therefore, we conclude that for any $\mathbf{i} \in \Gamma_1(r)$,

$$I_\mu\Big(q, \frac{r}{r_{\mathbf{i}}}\Big) = \int_{K_C} \mu\Big(B\Big(x, \frac{r}{r_{\mathbf{i}}}\Big)\Big)^{q-1} \mathrm{d}\mu(x) = 1.$$

Consequently, for any $q \geq 1$,

$$I_\mu(q,r) \geq \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q + p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r}{r_{\mathbf{i}}}\Big). \tag{3.6}$$

Proceeding as in the proof for $q \geq 1$, it is easy to check that for any $q \leq 1$ we have

$$I_\mu(q,r) \leq \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q + p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r}{r_{\mathbf{i}}}\Big). \tag{3.7}$$

(a) Consider $q \geq 1$. On one hand, it follows that

$$\begin{aligned} I_\mu(q,r) &\geq \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q + p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r}{r_{\mathbf{i}}}\Big) && \text{(using (3.6))} \\ &\geq \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q \\ &\geq C_{\min} r^{-\beta(q)} && \text{(using Lemma 3.2).} \end{aligned}$$

By the definition of $\underline{\tau}_\mu(q)$ in (1.2), we derive that $\underline{\tau}_\mu(q) \geq \beta(q)$. On the other hand, it can easily be seen that $I_\mu(q,r) \geq p^q I_\nu(q,r)$, which leads to the conclusion that $\underline{\tau}_\mu(q) \geq \underline{\tau}_\nu(q)$ and $\overline{\tau}_\mu(q) \geq \overline{\tau}_\nu(q)$. We have thus proved that

$$\begin{aligned} \underline{\tau}_\mu(q) &\geq \max\{\beta(q),\ \underline{\tau}_\nu(q)\}, \\ \overline{\tau}_\mu(q) &\geq \max\{\beta(q),\ \overline{\tau}_\nu(q)\}. \end{aligned}$$

(b) Consider $q \leq 1$. It follows from the definition of $\overline{\tau}_\nu(q)$ in (1.3) that there exists a positive constant c_1 such that for any $\mathbf{i} \in \Gamma_2(r)$, i.e. $\frac{r}{r_{\mathbf{i}}} < \operatorname{diam} K_C$, we have

$$I_\nu\Big(q, \frac{r}{r_{\mathbf{i}}}\Big) \leq c_1 \Big(\frac{r}{r_{\mathbf{i}}}\Big)^{-(\overline{\tau}_\nu(q)+\epsilon)}. \tag{3.8}$$

Thanks to Lemma 3.3, given $\epsilon > 0$, there exists $r_0 \in (0,1)$ such that for all $r \in (0, r_0)$ we have

$$\sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} < r^{-\epsilon}.$$

Combining (3.7), (3.8) and Lemma 3.2, we get

$$\begin{aligned} I_\mu(q,r) &\leq \sum_{\mathbf{i}\in\Gamma_1(r)} p_{\mathbf{i}}^q + c_1 p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q \Big(\frac{r}{r_{\mathbf{i}}}\Big)^{-(\overline{\tau}_\nu(q)+\epsilon)} \\ &\leq C_{\max} r^{-\beta(q)} + c_1 p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\overline{\tau}_\nu(q)+\epsilon} r^{-(\overline{\tau}_\nu(q)+\epsilon)}. \end{aligned}$$

If $\overline{\tau}_\nu(q) \geq \beta(q)$, then

$$\sum_i p_i^q r_i^{\overline{\tau}_\nu(q)+\epsilon} < \sum_i p_i^q r_i^{\beta(q)} = 1.$$

Evidently, the series

$$\sum_{k=0}^{\infty}\Big(\sum_{i} p_i^q r_i^{\overline{\tau}_\nu(q)+\epsilon}\Big)^k$$

is convergent, and we denote the limit by c_2. Consequently, we derive that

$$I_\mu(q,r) \le C_{\max} r^{-\beta(q)} + c_1 c_2 p^q r^{-(\overline{\tau}_\nu(q)+\epsilon)} \le (C_{\max} + c_1 c_2 p^q) r^{-(\overline{\tau}_\nu(q)+\epsilon)}.$$

Hence, we are led to $\overline{\tau}_\mu(q) \le \overline{\tau}_\nu(q) + \epsilon$. Letting $\epsilon \to 0$ yields that

$$\overline{\tau}_\mu(q) \le \overline{\tau}_\nu(q) = \max\{\overline{\tau}_\nu(q), \beta(q)\}.$$

If $\overline{\tau}_\nu(q) < \beta(q)$, since ϵ can be chosen arbitrarily small, we can assume that $\overline{\tau}_\nu(q) + \epsilon < \beta(q)$. It follows from (3.2), (3.7), (3.8) and Lemma 3.3 that

$$\begin{aligned} I_\mu(q,r) &\le C_{\max} r^{-\beta(q)} + c_1 p^q \sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} r_{\mathbf{i}}^{\overline{\tau}_\nu(q)+\epsilon-\beta(q)} r^{-(\overline{\tau}_\nu(q)+\epsilon)} \\ &\le C_{\max} r^{-\beta(q)} + c_1 p^q \Big(\frac{r}{\operatorname{diam} K_C}\Big)^{\overline{\tau}_\nu(q)+\epsilon-\beta(q)} r^{-(\overline{\tau}_\nu(q)+2\epsilon)} \\ &\le C_0 r^{-(\beta(q)+\epsilon)}, \end{aligned}$$

where

$$C_0 = C_{\max} + c_1 p^q (\operatorname{diam} K_C)^{\beta(q)-(\overline{\tau}_\nu(q)+\epsilon)}.$$

Thus, we deduce $\overline{\tau}_\mu(q) \le \beta(q) + \epsilon$, and letting $\epsilon \to 0$ yields that

$$\overline{\tau}_\mu(q) \le \beta(q) = \max\{\overline{\tau}_\nu(q), \beta(q)\}.$$

It remains to prove that

$$\underline{\tau}_\mu(q) \le \max\{\beta(q),\ \underline{\tau}_\nu(q),\ \underline{\tau}_\nu(q) + \beta(q) - (1-q)\dim_L \nu\}.$$

Fix $\epsilon > 0$. According to Lemma 3.3, there exists $r_0 \in (0,1)$ such that for any $r \in (0, r_0)$,

$$\sum_{\mathbf{i}\in\Gamma_3(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} < r^{-\epsilon}.$$

By the definition of $\underline{\tau}_\nu(q)$ in (1.2), there exists a sequence $\{r_m\}_m \searrow 0$ such that

$$I_\nu(q, r_m) \le r_m^{-(\underline{\tau}_\nu(q)+\epsilon)}. \tag{3.9}$$

We assume m large enough such that $0 < r_m < r_0$. Applying Lemma 3.2 to (3.7), we have

$$\begin{aligned} I_\mu(q, r_m) &\le \sum_{\mathbf{i}\in\Gamma_1(r_m)} p_{\mathbf{i}}^q + p^q \sum_{\mathbf{i}\in\Gamma_2(r_m)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r_m}{r_{\mathbf{i}}}\Big) \\ &\le C_{\max} r_m^{-\beta(q)} + p^q \sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r_m}{r_{\mathbf{i}}}\Big) + p^q \sum_{\mathbf{i}\in\Gamma_2(r_m)\setminus\Gamma_3(r_m)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r_m}{r_{\mathbf{i}}}\Big). \end{aligned} \tag{3.10}$$

For any $\mathbf{i} \in \Gamma_2(r_m) \setminus \Gamma_3(r_m)$, it follows that

$$\frac{r_m}{\operatorname{diam} K_C} < r_{\mathbf{i}} \le \frac{r_m}{\operatorname{diam} C}, \tag{3.11}$$

that is,

$$\operatorname{diam} C \le \frac{r_m}{r_{\mathbf{i}}} < \operatorname{diam} K_C. \tag{3.12}$$

From (3.11), we obtain

$$r_{\mathbf{i}}^{-\beta(q)} \le \max\Big\{\Big(\frac{r_m}{\operatorname{diam} K_C}\Big)^{-\beta(q)}, \Big(\frac{r_m}{\operatorname{diam} C}\Big)^{-\beta(q)}\Big\} \le C_{\max} r_m^{-\beta(q)}. \tag{3.13}$$

Moreover, according to (3.12), it is easy to check that

$$I_\nu(q, \frac{r_m}{r_{\mathbf{i}}}) = \int_C \nu\Big(B\Big(x, \frac{r_m}{r_{\mathbf{i}}}\Big)\Big)^{q-1} \, d\nu(x) = 1. \tag{3.14}$$

Based on the above argument, we observe that

$$\begin{aligned}
\sum_{\mathbf{i}\in\Gamma_2(r_m)\setminus\Gamma_3(r_m)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r_m}{r_{\mathbf{i}}}\Big) &= \sum_{\mathbf{i}\in\Gamma_2(r_m)\setminus\Gamma_3(r_m)} p_{\mathbf{i}}^q && \text{(using (3.14))} \\
&= \sum_{\mathbf{i}\in\Gamma_2(r_m)\setminus\Gamma_3(r_m)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} r_{\mathbf{i}}^{-\beta(q)} \\
&\le C_{\max} r_m^{-\beta(q)} \cdot \Big(\sum_{\mathbf{i}\in\Gamma_2(r_m)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)}\Big) && \text{(using (3.13))} \\
&\le C_{\max} r_m^{-(\beta(q)+\epsilon)} && \text{(using Lemma 3.3).}
\end{aligned} \tag{3.15}$$

For any $\mathbf{i} \in \Gamma_3(r_m)$, it follows from (3.3) that $\frac{r_m}{r_{\mathbf{i}}} < \operatorname{diam} C$. Due to the definition of $\dim_L \nu$ in (2.5), for any $s < \dim_L \nu$, there exists $c_3 > 0$ such that for any $\mathbf{i} \in \Gamma_3(r_m)$ and $x \in C$,

$$\nu\Big(B\Big(x, \frac{r_m}{r_{\mathbf{i}}}\Big)\Big) \ge \frac{c_3 \nu(B(x, r_m))}{r_{\mathbf{i}}^s}.$$

Clearly,

$$\begin{aligned}
I_\nu(q, \frac{r_m}{r_{\mathbf{i}}}) &= \int_C \nu\Big(B\Big(x, \frac{r_m}{r_{\mathbf{i}}}\Big)\Big)^{q-1} \, d\nu(x) \\
&\le c_3^{q-1} r_{\mathbf{i}}^{s(1-q)} \int_C \nu(B(x, r_m))^{q-1} \, d\nu(x) \\
&= c_3^{q-1} r_{\mathbf{i}}^{s(1-q)} I_\nu(q, r_m) \\
&\le c_3^{q-1} r_{\mathbf{i}}^{s(1-q)} r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \quad \text{(using (3.9)).}
\end{aligned} \tag{3.16}$$

If $(1-q)\dim_L \nu > \beta(q)$, since s can be chosen arbitrarily closed to $\dim_L \nu$, we can assume $s(1-q) > \beta(q)$. Obviously,

$$\sum_i p_i^q r_i^{s(1-q)} < \sum_i p_i^q r_i^{\beta(q)} = 1.$$

Write

$$c_4 = \sum_{k=0}^{\infty} \Big(\sum_i p_i^q r_i^{s(1-q)}\Big)^k = \sum_{\mathbf{i}\in\Sigma^*} p_{\mathbf{i}}^q r_{\mathbf{i}}^{s(1-q)}. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\begin{aligned}
\sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q I_\nu\Big(q, \frac{r_m}{r_{\mathbf{i}}}\Big) &\le \sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q c_3^{q-1} r_{\mathbf{i}}^{s(1-q)} r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \\
&\le c_3^{q-1} r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \cdot \Big(\sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{s(1-q)}\Big) \\
&\le c_4 c_3^{q-1} r_m^{-(\underline{\tau}_\nu(q)+\epsilon)}.
\end{aligned} \tag{3.18}$$

Using (3.15) and (3.18) in (3.10), we have

$$I_\nu(q, r_m) \le C_{\max} r_m^{-\beta(q)} + c_4 c_3^{q-1} p^q r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} + C_{\max} p^q r_m^{-(\beta(q)+\epsilon)}.$$

This implies that

$$\underline{\tau}_\mu(q) \le \max\{\underline{\tau}_\nu(q) + \epsilon, \beta(q) + \epsilon\}.$$

Letting $\epsilon \to 0$ yields that

$$\underline{\tau}_\mu(q) \le \max\{\beta(q), \ \underline{\tau}_\nu(q)\}.$$

We have

$$\underline{\tau}_\mu(q) \le \max\{\beta(q),\ \underline{\tau}_\nu(q),\ \underline{\tau}_\nu(q) - (1-q)\dim_L \nu + \beta(q)\}.$$

If $(1-q)\dim_L \nu \le \beta(q)$, then for any $s < \dim_L \nu$ we have $s(1-q) < \beta(q)$. It follows that

$$\begin{aligned}\sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{s(1-q)} &= \sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} r_{\mathbf{i}}^{s(1-q)-\beta(q)} \\ &\le \Big(\frac{r_m}{\operatorname{diam} C}\Big)^{s(1-q)-\beta(q)} \cdot \Big(\sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)}\Big) \\ &\le (\operatorname{diam} C)^{-s(1-q)+\beta(q)} \cdot r_m^{s(1-q)-\beta(q)-\epsilon} \quad \text{(using Lemma 3.3).}\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} I_\nu(q, r_m) &\le C_{\max} r_m^{-\beta(q)} + c_3^{q-1} p^q r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \cdot \Big(\sum_{\mathbf{i}\in\Gamma_3(r_m)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{s(1-q)}\Big) + C_{\max} p^q r_m^{-(\beta(q)+\epsilon)} \\ &\le C_{\max} r_m^{-\beta(q)} + c_3^{q-1} p^q (\operatorname{diam} C)^{\beta(q)-s(1-q)} r_m^{s(1-q)-\beta(q)-(\underline{\tau}_\nu(q)+2\epsilon)} + C_{\max} p^q r_m^{-(\beta(q)+\epsilon)}.\end{aligned}$$

This infers that

$$\underline{\tau}_\mu(q) \le \max\{\beta(q)+\epsilon,\ \underline{\tau}_\nu(q) + 2\epsilon - s(1-q) + \beta(q)\}.$$

Evidently, we derive that

$$\underline{\tau}_\mu(q) \le \max\{\beta(q),\ \underline{\tau}_\nu(q),\ \underline{\tau}_\nu(q) - (1-q)\dim_L \nu + \beta(q)\}.$$

It is now obvious that the theorem holds. □

Now, we show Theorem 2.8. Fix a positive integer n. Recall that for any $\mathbf{i} \in \bigcup_{k=0}^n \Sigma^k$ we write

$$M_{\mathbf{i}}^n = \begin{cases} S_{\mathbf{i}} K_C, & |\mathbf{i}| = n, \\ S_{\mathbf{i}} C, & 0 < |\mathbf{i}| < n, \\ C, & \mathbf{i} = \omega. \end{cases}$$

Set

$$\mathcal{A}_n = \Big\{\Delta \subseteq \bigcup_{k=0}^n \Sigma^k : \bigcap_{\mathbf{i}\in\Delta} M_{\mathbf{i}}^n \ne \emptyset\Big\}.$$

For any $\Delta \in \mathcal{A}_n$, we define

$$\Phi_\Delta(s) = \sum_{\mathbf{i}\in\Delta, |\mathbf{i}|=n} p_{\mathbf{i}} r_{\mathbf{i}}^s + p \sum_{\mathbf{i}\in\Delta, |\mathbf{i}|<n} p_{\mathbf{i}} r_{\mathbf{i}}^s,$$

and we write the unique solution of $\Phi_\Delta(s) = 1$ as $s(\Delta)$. Let

$$s_n = \sup_{\Delta\in\mathcal{A}_n} s(\Delta).$$

Denote the distance between two points $x, y \in \mathbb{R}^d$ by $|x-y|$. Write the distance between a point $x \in \mathbb{R}^d$ and a compact set $A \subseteq \mathbb{R}^d$ as

$$\operatorname{dist}(x, A) = \inf\{|x-y| : y \in A\},$$

and the distance between two compact sets $A, B \subseteq \mathbb{R}^d$ as

$$\operatorname{dist}(A, B) = \inf\{|x-y| : x \in A,\ y \in B\}.$$

Moreover, we define $\rho_n : K_C \to \mathbb{R}$ by

$$\rho_n(x) = \sup_{\Delta\in\mathcal{A}_n} \min_{\mathbf{i}\in\bigcup_{k=0}^n \Sigma^k\setminus\Delta} \operatorname{dist}(x, M_{\mathbf{i}}^n)$$

and let

$$\delta_n = \inf_{x\in K_C} \rho_n(x).$$

Lemma 3.5. *Assume that*

$$\bigcap_{|\mathbf{i}|\le n} M_{\mathbf{i}}^n = \emptyset.$$

Then $\delta_n > 0$.

Proof. We first show that $\rho_n(x) > 0$ for all $x \in K_C$. Set

$$\Delta_0 = \Big\{\mathbf{i} \in \bigcup_{k=0}^{n} \Sigma^k : x \in M_i^n\Big\}.$$

Since $x \in \bigcap_{\mathbf{i}\in\Delta_0} M_{\mathbf{i}}^n$, we have $\Delta_0 \in \mathcal{A}_n$. It follows from

$$\bigcap_{|\mathbf{i}|\le n} M_{\mathbf{i}}^n = \emptyset$$

that

$$\bigcup_{k=0}^{n} \Sigma^k \setminus \Delta_0 \ne \emptyset.$$

Moreover, for any $\mathbf{i} \in \bigcup_{k=0}^n \Sigma^k \setminus \Delta_0$, it is clear that $x \notin M_{\mathbf{i}}^n$. Thus,

$$\rho_n(x) \ge \min_{\mathbf{i}\in\bigcup_{k=0}^n \Sigma^k\setminus\Delta_0} \operatorname{dist}(x, M_{\mathbf{i}}^n) > 0.$$

We claim that $\rho_n(x)$ is continuous. It suffices to show that for any $x, y \in \mathbb{R}^d$,

$$|\rho_n(x) - \rho_n(y)| \le |x - y|. \tag{3.19}$$

For any $\Delta \in \mathcal{A}_n$, there exists $\mathbf{j}_0 \in \bigcup_{k=0}^n \Sigma^k \setminus \Delta$ such that

$$\operatorname{dist}(y, M_{\mathbf{j}_0}^n) = \min_{\mathbf{j}\in\bigcup_{k=0}^n \Sigma^k\setminus\Delta} \operatorname{dist}(y, M_{\mathbf{j}}^n) \le \rho_n(y). \tag{3.20}$$

Clearly,

$$\min_{\mathbf{i}\in\bigcup_{k=0}^n \Sigma^k\setminus\Delta} \operatorname{dist}(x, M_{\mathbf{i}}^n) \le \operatorname{dist}(x, M_{\mathbf{j}_0}^n). \tag{3.21}$$

Moreover, according to the compactness of $M_{\mathbf{i}}^n$, it follows that there exist $x_0, y_0 \in M_{\mathbf{i}}^n$ such that

$$\operatorname{dist}(x, M_{\mathbf{i}}^n) = |x - x_0|$$

and

$$\operatorname{dist}(y, M_{\mathbf{i}}^n) = |y - y_0|.$$

Obviously,

$$\operatorname{dist}(x, M_{\mathbf{i}}^n) - \operatorname{dist}(y, M_{\mathbf{i}}^n) = |x - x_0| - |y - y_0| \le |x - y|. \tag{3.22}$$

Combining (3.20)–(3.22) gives that

$$\begin{aligned}
\min_{\mathbf{i}\in\bigcup_{k=0}^n \Sigma^k\setminus\Delta} \operatorname{dist}(x, M_{\mathbf{i}}^n) - \rho_n(y) &\le \min_{\mathbf{i}\in\bigcup_{k=0}^n \Sigma^k\setminus\Delta} \operatorname{dist}(x, M_{\mathbf{i}}^n) - \min_{\mathbf{j}\in\bigcup_{k=0}^n \Sigma^k\setminus\Delta} \operatorname{dist}(y, M_{\mathbf{j}}^n) \\
&\le \operatorname{dist}(x, M_{\mathbf{j}_0}^n) - \operatorname{dist}(y, M_{\mathbf{j}_0}^n) \\
&\le |x - y|.
\end{aligned}$$

Taking the supremum over all $\Delta \in \mathcal{A}_n$ leads to

$$\rho_n(x) - \rho_n(y) \le |x - y|.$$

Owing to the arbitrariness of x and y, we therefore conclude (3.19) as claimed. Finally, based on the above argument, there exists $x_0 \in K_C$ such that

$$\delta_n = \inf_{x\in K_C} \rho_n(x) = \rho_n(x_0) > 0.$$

This completes the proof. □

Let ν be a probability measure. For any $r > 0$, we set

$$I_\nu(r) = \sup_{x \in \operatorname{spt} \nu} \nu(B(x, r))$$

and

$$I_\nu^{\mathbb{R}^d}(r) = \sup_{x \in \mathbb{R}^d} \nu(B(x, r)).$$

Recall that the lower ∞-th Rényi dimension of ν is defined by

$$\underline{D}_\nu(\infty) = \liminf_{r \to 0} \frac{\log I_\nu(r)}{\log r}.$$

Lemma 3.6. *Let the setting and notation be as above. The following statements hold:*

(i) *For any $r > 0$, we have*

$$I_\nu^{\mathbb{R}^d}\Big(\frac{r}{2}\Big) \le I_\nu(r) \le I_\nu^{\mathbb{R}^d}(r).$$

(ii) *We have the following equivalent definition:*

$$\underline{D}_\nu(\infty) = \liminf_{r \to 0} \frac{\log I_\nu^{\mathbb{R}^d}(r)}{\log r}.$$

Proof. (a) For any $r > 0$, without loss of generality, we assume that $I_\nu^{\mathbb{R}^d}(\frac{r}{2}) > 0$. For any $x \in \mathbb{R}^d$ with

$$\nu\Big(B\Big(x, \frac{r}{2}\Big)\Big) > 0,$$

there exists $x_0 \in \operatorname{spt} \nu \cap B(x, \frac{r}{2})$ such that

$$\nu\Big(B\Big(x, \frac{r}{2}\Big)\Big) \le \nu(B(x_0, r)) \le I_\nu(r).$$

Thus, we conclude that $I_\nu^{\mathbb{R}^d}(\frac{r}{2}) \le I_\nu(r)$. Moreover, it is well known that $I_\nu(r) \le I_\nu^{\mathbb{R}^d}(r)$.

(b) It is easy to see that

$$\frac{\log I_\nu^{\mathbb{R}^d}(r)}{\log r} \le \frac{\log I_\nu(r)}{\log r} \le \frac{\log I_\nu^{\mathbb{R}^d}(\frac{r}{2})}{\log \frac{r}{2} + \log 2}.$$

Letting $r \to 0$, we obtain the desired result. □

Lemma 3.7. *Let n be the smallest integer satisfying*

$$\bigcap_{|\mathbf{i}| \le n} M_{\mathbf{i}}^n = \emptyset.$$

For any $0 < r < \delta_n$, we have

$$I_\mu(r) \le \max_{\Delta \in \mathcal{A}_n} \Big\{ \sum_{\substack{\mathbf{i} \in \Delta \\ |\mathbf{i}| = n}} p_{\mathbf{i}} M_\mu\Big(\frac{r}{r_{\mathbf{i}}}\Big) + p \sum_{\substack{\mathbf{i} \in \Delta \\ |\mathbf{i}| < n}} p_{\mathbf{i}} I_\nu^{\mathbb{R}^d}\Big(\frac{r}{r_{\mathbf{i}}}\Big) \Big\}.$$

Proof. Consider $x \in K_C$. Denote the cardinality of a set A by $\#A$. Since $\#\mathcal{A}_n$ is finite, we conclude that there exists $\Delta_x \in \mathcal{A}_n$ such that

$$\rho_n(x) = \min_{\mathbf{j} \in \bigcup_{k=0}^n \Sigma^k \setminus \Delta_x} \operatorname{dist}(x, M_{\mathbf{j}}).$$

It is easy to check that for any $0 < r < \delta_n \le \rho_n(x)$,

$$\{\mathbf{i} : B(x, r) \cap M_{\mathbf{i}}^n \ne \emptyset\} \subseteq \Delta_x. \tag{3.23}$$

Indeed, for any $\mathbf{i} \in \bigcup_{k=0}^n \Sigma^k \setminus \Delta_x$,

$$\operatorname{dist}(x, M_{\mathbf{i}}^n) \ge \min_{\mathbf{j} \in \bigcup_{k=0}^n \Sigma^k \setminus \Delta_x} \operatorname{dist}(x, M_{\mathbf{j}}^n) = \rho_n(x) \ge \delta_n > r,$$

which infers that $B(x,r)\cap M_{\mathbf{i}}^n=\emptyset$. Clearly, for any $x\in K_C$, we derive that

$$\begin{aligned}\mu(B(x,r)) &= \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}\mu\circ S_{\mathbf{i}}^{-1}(B(x,r)) + p\sum_{|\mathbf{i}|<n} p_{\mathbf{i}}\nu\circ S_{\mathbf{i}}^{-1}(B(x,r)) \qquad \text{(using (2.1))}\\
&= \sum_{\substack{\mathbf{i}\in\Delta_x\\|\mathbf{i}|=n}} p_{\mathbf{i}}\mu\circ S_{\mathbf{i}}^{-1}(B(x,r)) + p\sum_{\substack{\mathbf{i}\in\Delta_x\\|\mathbf{i}|<n}} p_{\mathbf{i}}\nu\circ S_{\mathbf{i}}^{-1}(B(x,r)) \qquad \text{(using (3.23))}\\
&= \sum_{\substack{\mathbf{i}\in\Delta_x\\|\mathbf{i}|=n}} p_{\mathbf{i}}\mu\Big(B\Big(S_{\mathbf{i}}^{-1}x,\frac{r}{r_{\mathbf{i}}}\Big)\Big) + p\sum_{\substack{\mathbf{i}\in\Delta_x\\|\mathbf{i}|<n}} p_{\mathbf{i}}\nu\Big(B\Big(S_{\mathbf{i}}^{-1}x,\frac{r}{r_{\mathbf{i}}}\Big)\Big)\\
&\le \sum_{\substack{\mathbf{i}\in\Delta_x\\|\mathbf{i}|=n}} p_{\mathbf{i}}I_\mu\Big(\frac{r}{r_{\mathbf{i}}}\Big) + p\sum_{\substack{\mathbf{i}\in\Delta_x\\|\mathbf{i}|<n}} p_{\mathbf{i}}I_\nu^{\mathbb{R}^d}\Big(\frac{r}{r_{\mathbf{i}}}\Big)\\
&\le \max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}}I_\mu\Big(\frac{r}{r_{\mathbf{i}}}\Big) + p\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}}I_\nu^{\mathbb{R}^d}\Big(\frac{r}{r_{\mathbf{i}}}\Big)\Big\}.\end{aligned}$$

Taking the supremum over all $x\in K_C$ implies that

$$M_\mu(r)\le \max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}}I_\mu\Big(\frac{r}{r_{\mathbf{i}}}\Big) + p\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}}I_\nu^{\mathbb{R}^d}\Big(\frac{r}{r_{\mathbf{i}}}\Big)\Big\}$$

as desired. □

Proof of Theorem 2.8. Since n is the smallest integer such that

$$\bigcap_{|\mathbf{i}|\le n} M_{\mathbf{i}}^n=\emptyset,$$

we have $\delta_n>0$ from Lemma 3.5. Let $t>\max\{s_n,-\underline{D}_\nu(\infty)\}$. It follows from Lemma 3.6 that there exists $c_1>0$ such that for any $0<r<\delta_n$ and $|\mathbf{i}|<n$, i.e. $0<\frac{r}{r_{\mathbf{i}}}<\delta_n/r_{\min}^n$, we have

$$I_\nu^{\mathbb{R}^d}\Big(\frac{r}{r_{\mathbf{i}}}\Big)\le c_1\Big(\frac{r}{r_{\mathbf{i}}}\Big)^{-t}. \tag{3.24}$$

It follows that for any $0<r<\delta_n$,

$$\begin{aligned}I_\mu(r) &\le \max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}}I_\mu\Big(\frac{r}{r_{\mathbf{i}}}\Big) + p\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}}I_\nu^{\mathbb{R}^d}\Big(\frac{r}{r_{\mathbf{i}}}\Big)\Big\} \qquad \text{(using Lemma 3.7)}\\
&\le \max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}}I_\mu\Big(\frac{r}{r_{\mathbf{i}}}\Big) + c_1p\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}}r_{\mathbf{i}}^t r^{-t}\Big\} \qquad \text{(using (3.24)).}\end{aligned}\tag{3.25}$$

Let

$$c_2=\frac{I_\mu(\delta_n)}{\min\{(r_{\min}^n\delta_n)^{-t},\delta_n^{-t}\}}.$$

Thus, for any $r\in[r_{\min}^n\delta_n,\delta_n]$, we have

$$I_\mu(r)\le I_\mu(\delta_n)\le c_2r^{-t}.$$

Write $c_0=\max\{c_1,c_2\}$. We assume that for some $k\in\mathbb{N}$ the inequality

$$I_\mu(r)\le c_0r^{-t} \tag{3.26}$$

holds for any $r\in[r_{\min}^n r_{\max}^{kn}\delta_n,\delta_n]$. In particular, it obviously holds when $k=0$. We now show that (3.26) holds for any

$$r\in[r_{\min}^n r_{\max}^{(k+1)n}\delta_n,\delta_n].$$

For simplicity, we only need to verify the case of

$$r\in[r_{\min}^n r_{\max}^{(k+1)n}\delta_n, r_{\min}^n r_{\max}^{kn}\delta_n],$$

that is, for any $\mathbf{i}\in\Sigma^n$,

$$\frac{r}{r_{\mathbf{i}}}\in[r_{\min}^n r_{\max}^{kn}\delta_n,\delta_n].$$

It follows that

$$\begin{aligned} I_\mu(r) &\le \max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}} I_\mu\Big(\frac{r}{r_{\mathbf{i}}}\Big) + c_0 p \sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}} r_{\mathbf{i}}^t r^{-t}\Big\} \quad \text{(using (3.25))} \\ &\le \max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}} c_0\Big(\frac{r}{r_{\mathbf{i}}}\Big)^{-t} + c_0 p \sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}} r_{\mathbf{i}}^t r^{-t}\Big\} \\ &\le c_0 r^{-t} \max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}} r_{\mathbf{i}}^t + p \sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}} r_{\mathbf{i}}^t\Big\}. \end{aligned}$$

Since $t > s_n$, it is obvious to see that for any $\Delta \in \mathcal{A}_n$,

$$\Phi_\Delta(t) \le \Phi_\Delta(s_n) \le \Phi_\Delta(s(\Delta)) = 1.$$

Evidently,

$$\max_{\Delta\in\mathcal{A}_n}\Big\{\sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|=n}} p_{\mathbf{i}} r_{\mathbf{i}}^t + p \sum_{\substack{\mathbf{i}\in\Delta\\|\mathbf{i}|<n}} p_{\mathbf{i}} r_{\mathbf{i}}^t\Big\} \le 1.$$

Thus, we conclude that (3.26) holds for all

$$r \in [r_{\min}^n r_{\max}^{(k+1)n} \delta_n, r_{\min}^n r_{\max}^{kn} \delta_n].$$

By the inductive hypothesis, we have that (3.26) holds for all $r \in (0, \delta_n)$.

Consequently, for any $q \ge 1$ and $r \in (0, \delta_n)$,

$$I_\mu(q, r) = \int_{K_C} \mu(B(x, r))^{q-1}\, d\mu(x) \le I_\mu(r)^{q-1} \le c_0^{q-1} r^{-t(q-1)},$$

which leads to $\overline{\tau}_\mu(q) \le t(q-1)$. Letting $t \to \max\{s_n, -\underline{D}_\nu(\infty)\}$ gives the result that

$$\underline{\tau}_\mu(q) \le \overline{\tau}_\mu(q) \le \max\{(q-1)s_n, (1-q)\underline{D}_\nu(\infty)\}.$$

A similar argument deduces that for any $q \le 1$,

$$\overline{\tau}_\mu(q) \ge \underline{\tau}_\mu(q) \ge \max\{(q-1)s_n, (1-q)\underline{D}_\nu(\infty)\}.$$ □

4 IFS with some separation conditions

In this section, we study the L^q-spectra of the in-homogeneous self-similar measures generated by an IFS without some separation conditions. From Theorems 2.7, 2.9 and 2.10, it is easy to obtain Theorem 2.11. The main goal is to prove Theorems 2.9, 2.10 and 2.12.

Proof of Theorem 2.9. As $C \cap S_i K_C = \emptyset$ for all i, we have

$$\min_i \operatorname{dist}(C, S_i K_C) > 0.$$

We set $r_1 = \min_i \operatorname{dist}(C, S_i K_C)$ and consider $0 < r < r_1$. For any $x \in C$, it is apparent that $B(x, r) \cap S_i K_C = \emptyset$ for any $i \in \Sigma$, which implies that $\mu \circ S_i^{-1}(B(x, r)) = 0$ for any $i \in \Sigma$. Therefore, we have

$$\begin{aligned} I_\mu(q, r) &= \int_{K_C} \mu(B(x, r))^{q-1}\, d\mu(x) \\ &\ge p \int_C \mu(B(x, r))^{q-1}\, d\nu(x) \\ &= p^q \int_C \nu(B(x, r))^{q-1}\, d\nu(x) \\ &= p^q I_\nu(q, r). \end{aligned}$$

It turns out that

$$\underline{\tau}_\mu(q) \geq \underline{\tau}_\nu(q), \quad \overline{\tau}_\mu(q) \geq \overline{\tau}_\nu(q)$$

for all $q \in \mathbb{R}$. □

In the following, we assume that the IFS $\mathbf{I} = \{S_i\}_{i=1}^N$ satisfies (S1)–(S4). Our goal in this section is to prove Theorem 2.10. We need some lemmas.

Lemma 4.1. *Assume that the IFS* $\mathbf{I} = \{S_i\}_{i=1}^N$ *satisfies* (S1)–(S4). *Then the following assertions hold:*
(i) $\mu(\partial U) = 0$.
(ii) $K_C \subseteq \overline{U}$.
(iii) $\mu(S_{\mathbf{i}}U) = p_{\mathbf{i}}$ *for any* $\mathbf{i} \in \Sigma^*$.

Proof. (a) Consider $\mathbf{i} \in \Sigma^*$. It follows from (S1) and (S2) that

$$C \subseteq \overline{U} \subseteq S_{\mathbf{i}}^{-1}\overline{U}.$$

Then

$$C \cap S_{\mathbf{i}}^{-1}(\partial U) \subseteq C \cap \partial U.$$

Using (S4), we derive that $\nu \circ S_{\mathbf{i}}^{-1}(\partial U) = 0$. By the iterative formula (1.1), we derive that

$$\mu(\partial U) = \sum_{|\mathbf{i}|=n} p_{\mathbf{i}}\mu \circ S_{\mathbf{i}}^{-1}(\partial U) + p \sum_{|\mathbf{i}|<n} p_{\mathbf{i}}\nu \circ S_{\mathbf{i}}^{-1}(\partial U) \leq \sum_{|\mathbf{i}|=n} p_{\mathbf{i}} = (1-p)^n$$

for any $n \in \mathbb{N}$. Letting $n \to \infty$ gives $\mu(\partial U) = 0$.

(b) It is well known that

$$K_C = \overline{\bigcup_{\mathbf{i}\in\Sigma^*} S_{\mathbf{i}}(C)}.$$

According to (S1) and (S2), for any $\mathbf{i} \in \Sigma^*$, we have

$$S_{\mathbf{i}}C \subseteq S_{\mathbf{i}}\overline{U} \subseteq \overline{U}.$$

Evidently, we deduce that $K_C \subseteq \overline{U}$.

(c) Without loss of generality, we assume that $|\mathbf{i}| = n$. Consider $\mathbf{j} \neq \mathbf{i}$ with $|\mathbf{j}| = n$. Owing to (S2) and (S3), we observe that

$$S_{\mathbf{j}}U \cap S_{\mathbf{i}}U = \emptyset.$$

It follows from (a) and (b) that $\mu \circ S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}U) = 0$. Moreover, we consider $|\mathbf{j}| < n$. If $\mathbf{j}$ is not a prefix of $\mathbf{i}$, it is apparent to see that $\nu \circ S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}U) = 0$. If $\mathbf{j}$ is a prefix of $\mathbf{i}$, then there exists $i_0 \in \Sigma$ such that $S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}U) \subseteq S_{i_0}U$. Clearly, it follows from (S4) that

$$\nu \circ S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}U) \leq \nu(S_{i_0}U) = 0.$$

Due to the iterative formula (1.1) and the above discussion, we can conclude that

$$\mu(S_{\mathbf{i}}U) = \sum_{|\mathbf{j}|=n} p_{\mathbf{j}}\mu \circ S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}U) + p \sum_{|\mathbf{j}|<n} p_{\mathbf{j}}\nu \circ S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}U) = p_{\mathbf{i}}\mu \circ S_{\mathbf{i}}^{-1}(S_{\mathbf{i}}U) = p_{\mathbf{i}}.$$

The proof is finished. □

For any $0 < k < 1$ and $r > 0$, we write

$$\Omega_1(k, r) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} \leq \frac{r}{r_{\min} k \operatorname{diam} U} < r_{\mathbf{i}_-}\Big\}$$

and

$$\Omega_2(k, r) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} > \frac{r}{r_{\min} k \operatorname{diam} U}\Big\}.$$

Lemma 4.2. *For any* $0 < k < 1$, $r > 0$ *and* $q \in \mathbb{R}$, *we have*

$$\sum_{\mathbf{i}\in\Omega_1(k,r)} p_{\mathbf{i}}^q \geq \lambda r^{-\beta(q)},$$

where

$$\lambda = \min\{(k \operatorname{diam} U)^{-\beta(q)},\ (r_{\min} k \operatorname{diam} U)^{-\beta(q)}\}.$$

Proof. The proof of this result is quite similar to that given earlier in Lemma 3.1, and so it is omitted. □

Proof of Theorem 2.10. Given $0 < k < 1$, we set

$$kU = \{x \in U : \operatorname{dist}(x, \partial U) \geq k \operatorname{diam} U\}.$$

According to Lemma 4.1, we can easily find that $\mu(U) = 1$. Consequently, there exists $0 < k_0 < 1$ such that

$$\mu(k_0 U) \geq \frac{1}{2}. \tag{4.1}$$

Consider $r > 0$ and $q \leq 1$. For any $\mathbf{i} \in \Omega_1(k_0, r)$ and $x \in S_{\mathbf{i}}(K_C \cap k_0 U)$, it is easy to check that

$$\operatorname{dist}(x, \partial S_{\mathbf{i}} U) \geq r_{\mathbf{i}} k_0 \operatorname{diam} U > r.$$

This infers that

$$B(x, r) \subseteq S_{\mathbf{i}} U.$$

Consequently, for any $\mathbf{i} \in \Omega_1(k_0, r)$ and $x \in S_{\mathbf{i}}(K_C \cap k_0 U)$, we derive that

$$\mu(B(x, r))^{q-1} \geq \mu(S_{\mathbf{i}} U)^{q-1} \geq p_{\mathbf{i}}^{q-1} \quad \text{(using Lemma 4.1)}. \tag{4.2}$$

Clearly,

$$\begin{aligned}
I_\mu(q, r) &\geq \sum_{\mathbf{i} \in \Omega_1(k_0, r)} p_{\mathbf{i}} \int_{K_C} \mu(B(x, r))^{q-1} \, \mathrm{d}\mu \circ S_{\mathbf{i}}^{-1}(x) \\
&\geq \sum_{\mathbf{i} \in \Omega_1(k_0, r)} p_{\mathbf{i}} \int_{S_{\mathbf{i}}(K_C \cap k_0 U)} \mu(B(x, r))^{q-1} \, \mathrm{d}\mu \circ S_{\mathbf{i}}^{-1}(x) \\
&\geq \sum_{\mathbf{i} \in \Omega_1(k_0, r)} p_{\mathbf{i}}^q \mu(K_C \cap k_0 U) && \text{(using (4.2))} \\
&\geq \frac{1}{2} \sum_{\mathbf{i} \in \Omega_1(k_0, r)} p_{\mathbf{i}}^q && \text{(using (4.1))} \\
&\geq \frac{\lambda}{2} r^{-\beta(q)} && \text{(using Lemma 4.2)}.
\end{aligned}$$

This leads to the desired result that

$$\overline{\tau}_\mu(q) \geq \underline{\tau}_\mu(q) \geq \beta(q)$$

for all $q \leq 1$. □

In the final part of this section, we will prove Theorem 2.12. Firstly, we recall the upper and lower packing Rényi dimensions. We call a finite or countable family $\{B(x_k, r)\}_k$ of balls an *r-packing* of a set A if $B(x_k, r) \cap B(x_i, r) = \emptyset$ for all $i \neq k$ and $x_k \in A$ for all k. For any $q \in \mathbb{R}$ and $r > 0$, we set

$$M_\mu(q, r) = \sup\Big\{\sum_k \mu(B(x_k, r))^q : \{B(x_k, r)\}_k \text{ is an } r\text{-packing of } \operatorname{spt} \mu\Big\}.$$

Define respectively the *q-th lower packing Rényi dimension* $\underline{\tau}_\mu^p(q)$ and the *q-th upper packing Rényi dimension* $\overline{\tau}_\mu^p(q)$ of μ by

$$\underline{\tau}_\mu^p(q) = \liminf_{r \to 0} \frac{\log M_\mu(q, r)}{-\log r}$$

and

$$\overline{\tau}_\mu^p(q) = \limsup_{r \to 0} \frac{\log M_\mu(q, r)}{-\log r}.$$

Recall that

$$I_\mu(q, r) = \int_{\operatorname{spt} \mu} \mu(B(x, r))^{q-1} \, \mathrm{d}\mu(x).$$

Now, we study the relation between $I_\mu(q,r)$ and $M_\mu(q,r)$, and we prove that the q-th packing Rényi dimensions and L^q-spectra of μ are equivalent when $q \ge 1$.

Lemma 4.3. *Let the notation be as above. The following statements hold:*
(i) *There exists an integer P depending only on d such that for any $r > 0$ and $q \ge 1$,*

$$M_\mu\Big(q, \frac{r}{2}\Big) \le I_\mu(q,r) \le PM_\mu(q,3r).$$

(ii) *For any $q \ge 1$, we have*

$$\underline{\tau}_\mu(q) = \underline{\tau}^p_\mu(q), \quad \overline{\tau}_\mu(q) = \overline{\tau}^p_\mu(q).$$

Proof. (a) Fix $q \ge 1$ and $r > 0$. Let $\{B(x_k, \frac{r}{2})\}_{k\in\mathcal{I}}$ be an $\frac{r}{2}$-packing of $\operatorname{spt}\mu$. It is easy to see that for any $k \in \mathcal{I}$ and $x \in B(x_k, \frac{r}{2})$ we have

$$B\Big(x_k, \frac{r}{2}\Big) \subseteq B(x,r),$$

and thus

$$\mu\Big(B\Big(x_k, \frac{r}{2}\Big)\Big)^{q-1} \le \mu(B(x,r))^{q-1}.$$

Therefore,

$$\begin{aligned}\sum_{k\in\mathcal{I}} \mu\Big(B\Big(x_k, \frac{r}{2}\Big)\Big)^q &= \sum_{k\in\mathcal{I}} \int_{B(x_k,\frac{r}{2})} \mu\Big(B\Big(x_k, \frac{r}{2}\Big)\Big)^{q-1}\,\mathrm{d}\mu(x)\\ &\le \sum_{k\in\mathcal{I}} \int_{B(x_k,\frac{r}{2})} \mu(B(x,r))^{q-1}\,\mathrm{d}\mu(x)\\ &= \int_{\bigcup_k B(x_k,\frac{r}{2})} \mu(B(x,r))^{q-1}\,\mathrm{d}\mu(x)\\ &\le I_\mu(q,r).\end{aligned}$$

By the arbitrariness of the $\frac{r}{2}$-packing $\{B(x_k, \frac{r}{2})\}_{k\in\mathcal{I}}$, we derive that

$$M_\mu\Big(q, \frac{r}{2}\Big) \le I_\mu(q,r).$$

Now, we prove the second inequality. Let $\{B(x_k, r)\}_{k\in\mathcal{I}}$ be the largest possible collection of disjoint balls of radius r with centers in $\operatorname{spt}\mu$. Because of the compactness of $\operatorname{spt}\mu$, we have that $\mathcal{I}$ is a finite set. Obviously, $\{B(x_k, 2r)\}_{k\in\mathcal{I}}$ is a $2r$-covering of $\operatorname{spt}\mu$. As a matter of fact, if x belongs to $\operatorname{spt}\mu$, then x must be within distance r of one of the $B(x_k, r)$; otherwise, the ball of radius r centered at x can be added to form a larger collection of disjoint balls. Our first goal is to show that for each $k \in \mathcal{I}$,

$$\#\{i \in \mathcal{I} : B(x_i, 3r) \cap B(x_k, 3r) \ne \emptyset\} \le 9^d.$$

In other words, there are at most 9^d balls of radius $3r$ intersecting with $B(x_k, 3r)$. In fact, if

$$B(x_i, 3r) \cap B(x_k, 3r) \ne \emptyset,$$

then

$$B(x_i, r) \subseteq B(x_i, 3r) \subseteq B(x_k, 9r).$$

It follows from the disjointness of $\{B(x_i, r)\}_{i\in\mathcal{I}}$ that

$$\#\{i \in \mathcal{I} : B(x_i, 3r) \cap B(x_k, 3r) \ne \emptyset\} \le \Big(\frac{9r}{r}\Big)^d = 9^d. \tag{4.3}$$

For simplicity, we write $B_k = B(x_k, 3r)$ and $\mathcal{I} = \{1, \dots, m\}$, and we set $\mathcal{B} = \{B_1, \dots, B_m\}$.

We next remark that there exists an integer $P\,(\le 9^d + 1)$, depending only on d, such that there are families $\mathcal{B}_1, \dots, \mathcal{B}_P \subseteq \mathcal{B}$ satisfying that each $\mathcal{B}_j$ is disjoint and

$$\bigcup_{j=1}^{P} \mathcal{B}_j = \mathcal{B}.$$

Without loss of generality, we assume that $\#\mathcal{J} > 9^d + 1$. Let $B_{1,1} = B_1$ and then inductively choose $B_{1,j} = B_k$ for $j \geq 2$, where k is the smallest integer with

$$B_k \cap \bigcup_{i=1}^{j-1} B_{1,i} = \emptyset.$$

We continue this as long as possible getting a finite disjoint subfamily $\mathcal{B}_1 = \{B_{1,1}, \ldots, B_{1,m_1}\}$. If $\mathcal{B}_1 \neq \mathcal{B}$, we define first $B_{2,1} = B_k$, where k is the smallest integer for which $B_k \notin \mathcal{B}_1$. For $j \geq 2$, we define inductively $B_{2,j} = B_k$ with the smallest k such that $B_k \notin \mathcal{B}_1$ and

$$B_k \cap \bigcup_{i=1}^{j-1} B_{2,i} = \emptyset.$$

More generally, if $\bigcup_{l=1}^{s} \mathcal{B}_l \neq \mathcal{B}$, let $B_{s+1,1} = B_k$, where k is the smallest integer for which $B_k \notin \bigcup_{l=1}^{s} \mathcal{B}_l$. Again for $j \geq 2$, we define inductively $B_{s+1,j} = B_k$ with the smallest k such that $B_k \notin \bigcup_{l=1}^{s} \mathcal{B}_l$ and

$$B_k \cap \bigcup_{i=1}^{j-1} B_{s+1,i} = \emptyset.$$

With this process, we find a disjoint subfamily $\mathcal{B}_l = \{B_{l,1}, \ldots, B_{l,m_l}\}$. By this construction, we can find subfamilies $\mathcal{B}_1, \ldots, \mathcal{B}_P$ of $\mathcal{B}$ such that $\bigcup_{l=1}^{P} \mathcal{B}_l = \mathcal{B}$. If $P > 9^d + 1$, then for any $B_{P,i} \in \mathcal{B}_P$ and $l \in \{1, \ldots, 9^d + 1\}$ it follows from the construction of $\mathcal{B}_l$ that $B_{P,i} \cap B_{l,i_l} \neq \emptyset$ for some $i_l \in \{1, \ldots, m_l\}$. That is, $B_{P,i}$ intersects with at least $9^d + 1$ many balls in $\mathcal{B}$, which contradicts (4.3). Hence $P \leq 9^d + 1$ follows, as asserted .

Based on the above discussion, we derive

$$\begin{aligned}
I_\mu(q, r) &= \int\limits_{\bigcup_{k\in\mathcal{J}} B(x_k, 2r)} \mu(B(x, r))^{q-1}\, \mathrm{d}\mu(x) \\
&\leq \sum_{k\in\mathcal{J}} \int\limits_{B(x_k, 2r)} \mu(B(x, r))^{q-1}\, \mathrm{d}\mu(x) \\
&\leq \sum_{k\in\mathcal{J}} \int\limits_{B(x_k, 2r)} \mu(B_k)^{q-1}\, \mathrm{d}\mu(x) \\
&\leq \sum_{k\in\mathcal{J}} \mu(B_k)^q \\
&\leq \sum_{j=1}^{P} \sum_{B_k\in\mathcal{B}_j} \mu(B_k)^q \\
&\leq PM_\mu(q, 3r).
\end{aligned}$$

(b) This is now a direct consequence of (a). □

In the following, we assume that the COSC is satisfied and U is the open set used for the COSC.

Lemma 4.4. *Let the setting and notation be as above. We have the following properties:*
(i) *For any $\mathbf{i}, \mathbf{j} \in \Sigma^*$ with $\mathbf{i} \neq \mathbf{j}$, we have $S_{\mathbf{i}}C \cap S_{\mathbf{j}}C = \emptyset$.*
(ii) *For any $\mathbf{i} \in \Sigma^*$, we have $\mu(S_{\mathbf{i}}C) = pp_{\mathbf{i}}$.*
(iii) *For any $\mathbf{i} \in \Sigma^*$, we have $\mu(S_{\mathbf{i}}\overline{U}) = p_{\mathbf{i}}$.*

Proof. (a) Consider $\mathbf{i}, \mathbf{j} \in \Sigma^*$ with $\mathbf{i} \neq \mathbf{j}$. Without loss of generality, we assume that $|\mathbf{j}| \geq |\mathbf{i}|$. If $\mathbf{i}$ is a prefix of $\mathbf{j}$, it follows from the assumption of the COSC that for any $i \in \Sigma$ we have

$$S_i U \cap C = \emptyset, \quad S_i U \subseteq U.$$

Consequently,

$$C \cap S_{\mathbf{i}}^{-1} S_{\mathbf{j}} U = \emptyset,$$

that is, $S_{\mathbf{i}}C \cap S_{\mathbf{j}}U = \emptyset$. Since $C \subseteq U$, we have $S_{\mathbf{j}}C \subseteq S_{\mathbf{j}}U$. Evidently,

$$S_{\mathbf{i}}C \cap S_{\mathbf{j}}C = \emptyset.$$

If $\mathbf{i}$ is not a prefix of $\mathbf{j}$, then $S_{\mathbf{i}}U \cap S_{\mathbf{j}}U = \emptyset$. Moreover, by the fact that $S_{\mathbf{i}}C \subseteq S_{\mathbf{i}}U$ and $S_{\mathbf{j}}C \subseteq S_{\mathbf{j}}U$, we observe that $S_{\mathbf{i}}C \cap S_{\mathbf{j}}C = \emptyset$.

(b) From (a), we have $S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}C) \cap C = \emptyset$ for any $\mathbf{i}, \mathbf{j} \in \Sigma^*$ with $\mathbf{i} \neq \mathbf{j}$. According to (2.2), it is easy to see that for any $\mathbf{i} \in \Sigma^*$,

$$\begin{aligned}\mu(S_{\mathbf{i}}C) &= p \sum_{\mathbf{j}\in\Sigma^*} p_{\mathbf{j}} \nu \circ S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}C) \\ &= pp_{\mathbf{i}}\nu(C) + p \sum_{\substack{\mathbf{j}\in\Sigma^* \\ \mathbf{j}\neq\mathbf{i}}} p_{\mathbf{j}} \nu \circ S_{\mathbf{j}}^{-1}(S_{\mathbf{i}}C) \\ &= pp_{\mathbf{i}},\end{aligned}$$

where the final equality follows from (a).

(c) Consider $\mathbf{i} \in \Sigma^*$. As stated in (a), a routine analysis gives that $S_{\mathbf{j}}C \cap \partial S_{\mathbf{i}}U = \emptyset$ for any $\mathbf{j} \in \Sigma^*$. Moreover, it holds that

$$\Big(\bigcup_{\mathbf{j}\in\Sigma^*} S_{\mathbf{j}}C\Big) \cup \partial S_{\mathbf{i}}U \subseteq \overline{U}.$$

Clearly,

$$\begin{aligned}1 = \mu(\overline{U}) &\geq \sum_{\mathbf{j}\in\Sigma^*} \mu(S_{\mathbf{j}}C) + \mu(\partial S_{\mathbf{i}}U) \\ &= \sum_{k=0}^{\infty} \sum_{|\mathbf{j}|=k} pp_{\mathbf{j}} + \mu(\partial S_{\mathbf{i}}U) \quad \text{(using (b))} \\ &= 1 + \mu(\partial S_{\mathbf{i}}U).\end{aligned}$$

This infers that $\mu(\partial S_{\mathbf{i}}U) = 0$. Due to Lemma 4.1, we are led to the conclusion that

$$\mu(S_{\mathbf{i}}\overline{U}) = \mu(S_{\mathbf{i}}U) + \mu(\partial S_{\mathbf{i}}U) = p_{\mathbf{i}}.$$

We have completed the proof. □

Set

$$\kappa_0 = \min\{\operatorname{dist}(C, \partial U), \min_{1\leq i\leq N} \operatorname{dist}(C, S_i\overline{U})\}. \tag{4.4}$$

It is obvious that $\kappa_0 > 0$. For any $r \in (0, \kappa_0)$, we set

$$\mathcal{P}_1(r) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} \leq \frac{r}{\kappa_0} < r_{\mathbf{i}_-}\Big\}$$

and

$$\mathcal{P}_2(r) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} > \frac{r}{\kappa_0}\Big\}. \tag{4.5}$$

Lemma 4.5. *For any $r \in (0, \kappa_0)$, we have the following properties:*

(i) *For any $\mathbf{i} \in \mathcal{P}_1(r)$ and $\mathbf{j} \in \mathcal{P}_2(r)$, we have*

$$\operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}C) > r.$$

(ii) *There exists a positive constant K_1, depending on d and U, such that for any $\mathbf{i} \in \mathcal{P}_1(r)$,*

$$\#\{\mathbf{j} \in \mathcal{P}_1(r) : \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U}) < r\} \leq K_1.$$

(iii) *Write $\lambda_1 = K_1^{q+1}\kappa_0^{\beta(q)}$. We have*

$$M_\mu(q, r) \leq \sum_{j=1}^{N} p_j^q M_\mu\Big(q, \frac{r}{r_j}\Big) + p^q M_\nu(q, r) + \lambda_1 r^{-\beta(q)}.$$

Proof. (a) Consider $\mathbf{i} \in \mathcal{P}_1(r)$ and $\mathbf{j} \in \mathcal{P}_2(r)$. If $\mathbf{j}$ is a prefix of $\mathbf{i}$, i.e. $\mathbf{i} = (\mathbf{j}, \mathbf{k})$, where $\mathbf{k} \in \Sigma^*$, then

$$\begin{aligned}\operatorname{dist}(S_{\mathbf{j}}C, S_{\mathbf{i}}\overline{U}) &= r_{\mathbf{j}} \cdot \operatorname{dist}(C, S_{\mathbf{k}}\overline{U}) && (\text{since } \mathbf{i} = (\mathbf{j}, \mathbf{k}))\\ &\geq r_{\mathbf{j}} \cdot \operatorname{dist}(C, S_{\mathbf{k}|_1}\overline{U}) && (\text{since } S_{\mathbf{k}}\overline{U} \subseteq S_{\mathbf{k}|_1}\overline{U})\\ &> \frac{r}{\kappa_0} \cdot \kappa_0 && (\text{by (4.4) and (4.5)})\\ &= r.\end{aligned}$$

If $\mathbf{i}$ and $\mathbf{j}$ have different prefixes, then it follows from the COSC that

$$S_{\mathbf{j}}C \subseteq S_{\mathbf{j}}U, \quad S_{\mathbf{j}}U \cap S_{\mathbf{i}}U = \emptyset.$$

Therefore, we deduce that

$$\begin{aligned}\operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}C) &\geq \operatorname{dist}(S_{\mathbf{j}}C, \partial S_{\mathbf{j}}U)\\ &= r_{\mathbf{j}} \operatorname{dist}(C, \partial U)\\ &> \frac{r}{\kappa_0} \cdot \kappa_0 \quad (\text{by (4.4) and (4.5)})\\ &= r.\end{aligned}$$

(b) As U is an open set, we can assume further that U is contained in a ball of radius a, and U contains a ball of radius b. Consider $\mathbf{i} \in \mathcal{P}_1(r)$. It is easy to see that, for any $\mathbf{j} \in \mathcal{P}_1(r)$, $S_{\mathbf{j}}\overline{U}$ is contained in a ball of radius $\frac{ar}{\kappa_0}$ and it contains a ball of radius $\frac{br_{\min}r}{\kappa_0}$. Suppose there are K of $\mathbf{j}$ in $\mathcal{P}_1(r)$ such that

$$\operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U}) < r.$$

Then they are all contained in a ball of radius $(\frac{3a}{\kappa_0} + 1)r$ and each of them contains a ball of radius $\frac{br_{\min}r}{\kappa_0}$. Since the COSC is satisfied, the pieces $\{S_{\mathbf{j}}U\}_{\mathbf{j}\in\mathcal{P}_1(r)}$ are mutually disjoint. Summing up the volumes, we have

$$K\Big(\frac{br_{\min}r}{\kappa_0}\Big)^d \leq \Big(\frac{3a}{\kappa_0} + 1\Big)^d r^d.$$

The lemma follows by choosing $K_1 = (\frac{3a+\kappa_0}{br_{\min}})^d$.

(c) Let $\{B(x_k, r)\}_{k\in\mathcal{I}}$ be an arbitrary r-packing of K_C. It is straightforward to see that

$$x_k \in K_C \subseteq \Big(\bigcup_{\mathbf{i}\in\mathcal{P}_1(r)} S_{\mathbf{i}}\overline{U}\Big) \cup \Big(\bigcup_{\mathbf{i}\in\mathcal{P}_2(r)} S_{\mathbf{i}}C\Big) \tag{4.6}$$

for all $k \in \mathcal{I}$. Then

$$\sum_{k\in\mathcal{I}} \mu(B(x_k, r))^q \leq \sum_{\mathbf{i}\in\mathcal{P}_2(r)} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}C}} \mu(B(x_k, r))^q + \sum_{\mathbf{i}\in\mathcal{P}_1(r)} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}\overline{U}}} \mu(B(x_k, r))^q. \tag{4.7}$$

Consider the first summation of (4.7). Clearly,

$$\sum_{\mathbf{i}\in\mathcal{P}_2(r)} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}C}} \mu(B(x_k, r))^q \leq \sum_{j=1}^{N} \sum_{\substack{\mathbf{i}\in\mathcal{P}_2(r)\\ \mathbf{i}|_1=j}} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}C}} \mu(B(x_k, r))^q + \sum_{\substack{k\in\mathcal{I}\\ x_k\in C}} \mu(B(x_k, r))^q. \tag{4.8}$$

If $\mathbf{i} \in \mathcal{P}_2(r)$ with $\mathbf{i}|_1 = j \in \Sigma$, then

$$\begin{aligned}\operatorname{dist}(S_{\mathbf{i}}C, \partial S_jU) &\geq \operatorname{dist}(S_{\mathbf{i}}C, \partial S_{\mathbf{i}}U) && (\text{using } S_{\mathbf{i}}C \subseteq S_{\mathbf{i}}U \subseteq S_jU)\\ &= r_{\mathbf{i}} \cdot \operatorname{dist}(C, \partial U)\\ &> \frac{r}{\kappa_0} \cdot \kappa_0 && (\text{using (4.4) and (4.5)})\\ &= r.\end{aligned}$$

Evidently, it can easily be seen that for any $x_k \in S_{\mathbf{i}}C$,

$$B(x_k, r) \subseteq S_j U.$$

This implies that

$$\begin{aligned}\mu(B(x_k, r))^q &= \Big(\sum_{i=1}^{N} p_i \mu \circ S_i^{-1}(B(x_k, r)) + p\nu(B(x_k, r))\Big)^q \\ &= p_j^q \mu \circ S_j^{-1}(B(x_k, r))^q \\ &= p_j^q \mu\Big(B\Big(S_j^{-1}x_k, \frac{r}{r_j}\Big)\Big)^q. \end{aligned} \tag{4.9}$$

We next consider $x_k \in C$. Recall that

$$0 < r < \kappa_0 \leq \min_{1\leq j\leq N} \operatorname{dist}(C, S_j\overline{U}).$$

Therefore, for all $j \in \Sigma$, we get

$$B(x_k, r) \cap S_j\overline{U} = \emptyset.$$

Consequently,

$$\mu(B(x_k, r))^q = \Big(\sum_{i=1}^{N} p_i \mu \circ S_i^{-1}(B(x_k, r)) + p\nu(B(x_k, r))\Big)^q = p^q \nu(B(x_k, r))^q. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8), we derive

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{P}_2(r)} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}C}} \mu(B(x_k, r))^q &\leq \sum_{j=1}^{N} \sum_{\substack{\mathbf{i}\in\mathcal{P}_2(r)\\ \mathbf{i}|_1=j}} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}C}} p_j^q \mu\Big(B\Big(S_j^{-1}x_k, \frac{r}{r_j}\Big)\Big)^q + \sum_{\substack{k\in\mathcal{I}\\ x_k\in C}} p^q \nu(B(x_k, r))^q \\ &= \sum_{j=1}^{N} p_j^q \sum_{\substack{\mathbf{i}\in\mathcal{P}_2(r)\\ \mathbf{i}|_1=j}} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}C}} \mu\Big(B\Big(S_j^{-1}x_k, \frac{r}{r_j}\Big)\Big)^q + p^q \sum_{\substack{k\in\mathcal{I}\\ x_k\in C}} \nu(B(x_k, r))^q. \end{aligned}$$

It is easy to check that for any $j \in \Sigma$,

$$\Big\{B\Big(S_j^{-1}x_k, \frac{r}{r_j}\Big) : x_k \in S_{\mathbf{i}}C,\ k \in \mathcal{I},\ \text{where } \mathbf{i} \in \mathcal{P}_2(r) \text{ with } \mathbf{i}|_1 = j\Big\}$$

is an $\frac{r}{r_j}$-packing of K_C, and

$$\{B(x_k, r) : x_k \in C,\ k \in \mathcal{I}\}$$

is an r-packing of C. Clearly,

$$\sum_{\mathbf{i}\in\mathcal{P}_2(r)} \sum_{\substack{k\in\mathcal{I}\\ x_k\in S_{\mathbf{i}}C}} \mu(B(x_k, r))^q \leq \sum_{j=1}^{N} p_j^q M_\mu\Big(q, \frac{r}{r_j}\Big) + p^q M_\nu(q, r). \tag{4.11}$$

We next estimate $\mu(B(x_k, r))$ for $x_k \in S_{\mathbf{i}}\overline{U}$, $\mathbf{i} \in \mathcal{P}_1(r)$. By (4.6), we derive that

$$\mu(B(x_k, r)) \leq \sum_{\mathbf{j}\in\mathcal{P}_2(r)} \mu(B(x_k, r) \cap S_{\mathbf{j}}C) + \sum_{\mathbf{j}\in\mathcal{P}_1(r)} \mu(B(x_k, r) \cap S_{\mathbf{j}}\overline{U}).$$

According to (a), for any $\mathbf{j} \in \mathcal{P}_2(r)$, we have

$$\operatorname{dist}(x_k, S_{\mathbf{j}}C) \geq \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}C) > r,$$

which infers that

$$B(x_k, r) \cap S_{\mathbf{j}}C = \emptyset.$$

Hence, we obtain

$$\sum_{\mathbf{j}\in\mathcal{P}_2(r)} \mu(B(x_k, r)\cap S_{\mathbf{j}}C) = 0.$$

Moreover, it is straightforward to see that $\mu(B(x_k, r)\cap S_{\mathbf{j}}\overline{U}) > 0$ only if $\operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U}) < r$. It follows that

$$\mu(B(x_k, r)) \le \sum_{\substack{\mathbf{j}\in\mathcal{P}_1(r)\\ \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U})<r}} \mu(B(x_k, r)\cap S_{\mathbf{j}}\overline{U}). \tag{4.12}$$

Owing to the above argument, we are now in a position to consider the second summation of (4.7). Using inequality (4.12), we get

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathcal{P}_1(r)} \sum_{\substack{k\in\mathcal{J}\\ x_k\in S_{\mathbf{i}}\overline{U}}} \mu(B(x_k, r))^q &\le \sum_{\mathbf{i}\in\mathcal{P}_1(r)} \sum_{\substack{k\in\mathcal{J}\\ x_k\in S_{\mathbf{i}}\overline{U}}} \Big(\sum_{\substack{\mathbf{j}\in\mathcal{P}_1(r)\\ \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U})<r}} \mu(B(x_k, r)\cap S_{\mathbf{j}}\overline{U})\Big)^q \\
&\le \sum_{\mathbf{i}\in\mathcal{P}_1(r)} \Big(\sum_{\substack{k\in\mathcal{J}\\ x_k\in S_{\mathbf{i}}\overline{U}}} \sum_{\substack{\mathbf{j}\in\mathcal{P}_1(r)\\ \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U})<r}} \mu(B(x_k, r)\cap S_{\mathbf{j}}\overline{U})\Big)^q \\
&= \sum_{\mathbf{i}\in\mathcal{P}_1(r)} \Big(\sum_{\substack{\mathbf{j}\in\mathcal{P}_1(r)\\ \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U})<r}} \sum_{\substack{k\in\mathcal{J}\\ x_k\in S_{\mathbf{i}}\overline{U}}} \mu(B(x_k, r)\cap S_{\mathbf{j}}\overline{U})\Big)^q.
\end{aligned} \tag{4.13}$$

By means of the disjointness of $\{B(x_k, r)\}_{k\in\mathcal{J}}$, it turns out that

$$\begin{aligned}
\sum_{\substack{k\in\mathcal{J}\\ x_k\in S_{\mathbf{i}}\overline{U}}} \mu(B(x_k, r)\cap S_{\mathbf{j}}\overline{U}) &= \mu\Big(\bigcup_{\substack{k\in\mathcal{J}\\ x_k\in S_{\mathbf{i}}\overline{U}}} B(x_k, r)\cap S_{\mathbf{j}}\overline{U}\Big) \\
&\le \mu(S_{\mathbf{j}}\overline{U}) \\
&= p_{\mathbf{j}} \quad \text{(using Lemma 4.4).}
\end{aligned} \tag{4.14}$$

Furthermore, combining (4.13) and (4.14), we have

$$\begin{aligned}
\sum_{\mathbf{i}\in\mathcal{P}_1(r)} \sum_{\substack{k\in\mathcal{J}\\ x_k\in S_{\mathbf{i}}\overline{U}}} \mu(B(x_k, r))^q &\le \sum_{\mathbf{i}\in\mathcal{P}_1(r)} \Big(\sum_{\substack{\mathbf{j}\in\mathcal{P}_1(r)\\ \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U})<r}} p_{\mathbf{j}}\Big)^q \\
&\le K_1^q \sum_{\mathbf{i}\in\mathcal{P}_1(r)} \sum_{\substack{\mathbf{j}\in\mathcal{P}_1(r)\\ \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U})<r}} p_{\mathbf{j}}^q \quad \text{(using } (b))\\
&= K_1^q \sum_{\mathbf{j}\in\mathcal{P}_1(r)} \sum_{\substack{\mathbf{i}\in\mathcal{P}_1(r)\\ \operatorname{dist}(S_{\mathbf{i}}\overline{U}, S_{\mathbf{j}}\overline{U})<r}} p_{\mathbf{j}}^q \\
&\le K_1^{q+1} \sum_{\mathbf{j}\in\mathcal{P}_1(r)} p_{\mathbf{j}}^q \\
&\le \lambda_1 r^{-\beta(q)}.
\end{aligned} \tag{4.15}$$

If we plug (4.11) and (4.15) back into (4.7), then we derive

$$\sum_{k\in\mathcal{J}} \mu(B(x_k, r))^q \le \sum_{j=1}^{N} p_j^q M_\mu\Big(q, \frac{r}{r_j}\Big) + p^q M_\nu(q, r) + \lambda_1 r^{-\beta(q)}.$$

On account of the arbitrariness of $\{B(x_k, r)\}_{k\in\mathcal{J}}$, it turns that

$$M_\mu(q, r) \le \sum_{j=1}^{N} p_j^q M_\mu\Big(q, \frac{r}{r_j}\Big) + p^q M_\nu(q, r) + \lambda_1 r^{-\beta(q)}$$

as desired. □

Proof of Theorem 2.12. Fix $q \geq 1$. We first show that

$$\overline{\tau}_\mu(q) \leq \max\{\overline{\tau}_\nu(q), \beta(q)\}.$$

Let $t > \max\{\overline{\tau}_\nu(q), \beta(q)\}$. Then there exists $\kappa_1 \in (0, 1)$ such that for any $r \in (0, \kappa_1)$,

$$M_\nu(q, r) \leq r^{-t}. \tag{4.16}$$

Write $\kappa = \min\{\kappa_0, \kappa_1\}$ and consider $r \in (0, \kappa)$. As demonstrated in Lemma 4.5, we have

$$\begin{aligned} M_\mu(q, r) &\leq \sum_{j=1}^{N} p_j^q M_\mu\Big(q, \frac{r}{r_j}\Big) + p^q M_\nu(q, r) + \lambda_1 r^{-\beta(q)} \\ &\leq \sum_{j=1}^{N} p_j^q M_\mu\Big(q, \frac{r}{r_j}\Big) + \lambda_2 r^{-t} \quad \text{(using (4.16))}, \end{aligned} \tag{4.17}$$

where, for the ease of notation, we set $\lambda_2 = p^q + \lambda_1$. Choose a constant C_0 large enough such that

$$C_0 > \max\Big\{\frac{I_\mu(q, 2r_0)}{\min\{(r_{\min} r_0)^{-t}, r_0^{-t}\}}, \frac{\lambda_2}{1 - \sum_{j=1}^N p_j^q r_j^t}\Big\}.$$

It is easy to see that for any $r \in [r_{\min} r_0, r_0]$,

$$\begin{aligned} M_\mu(q, r) &\leq I_\mu(q, 2r) \quad \text{(using Lemma 4.3)} \\ &\leq I_\mu(q, 2r_0) \\ &= C_0 \min\{(r_{\min} r_0)^{-t}, r_0^{-t}\} \\ &\leq C_0 r^{-t}. \end{aligned}$$

Assume that

$$M_\mu(q, r) \leq C_0 r^{-t} \tag{4.18}$$

holds when $r \in [r_{\max}^n r_{\min} r_0, r_0]$ for some $n \in \mathbb{N}$. Now, we show that (4.18) holds for all $r \in [r_{\max}^{n+1} r_{\min} r_0, r_0]$. Apparently, we only need to consider $r \in [r_{\max}^{n+1} r_{\min} r_0, r_{\max}^n r_{\min} r_0]$, which infers that $\frac{r}{r_j} \in [r_{\max}^n r_{\min} r_0, r_0]$ for $j \in \Sigma$. We find that

$$\begin{aligned} M_\mu(q, r) &\leq \sum_{j=1}^{N} p_j^q C_0 \Big(\frac{r}{r_j}\Big)^{-t} + \lambda_2 r^{-t} \quad \text{(using (4.17) and (4.18))} \\ &= \Big(C_0 \sum_{j=1}^{N} p_j^q r_j^t + \lambda_2\Big) r^{-t} \\ &\leq C_0 r^{-t} \qquad \Big(\text{using } C_0 > \frac{\lambda_2}{1 - \sum_{j=1}^N p_j^q r_j^t}\Big). \end{aligned}$$

Therefore, the inductive hypothesis gives that (4.18) holds for all $r \in [0, r_0]$. Hence, we deduce $\overline{\tau}_\mu(q) \leq t$. Letting $t \to \max\{\overline{\tau}_\nu(q), \beta(q)\}$ yields that

$$\overline{\tau}_\mu(q) \leq \max\{\overline{\tau}_\nu(q), \beta(q)\}.$$

Thanks to Theorem 2.7, we are led to the result that

$$\overline{\tau}_\mu(q) = \{\overline{\tau}_\nu(q), \beta(q)\}$$

for all $q \geq 1$.

It remains to show that

$$\underline{\tau}_\mu(q) \leq \max\{\beta(q), \underline{\tau}_\nu(q), \underline{\tau}_\nu(q) + \beta(q) - (1 - q)\dim_A \nu\}.$$

Fix $\epsilon > 0$. It follows from the definition of $\underline{\tau}_\nu(q)$ in (1.2) that there exists a sequence $\{r_m\}_m \searrow 0$ such that

$$I_\nu(q, r_m) \le r_m^{-(\underline{\tau}_\nu(q)+\epsilon)}. \tag{4.19}$$

According to Lemma 3.3, there exists $0 < r_0 < 1$ such that for any $0 < r < r_0$ we have

$$\sum_{\mathbf{i}\in\Gamma_2(r)} p_{\mathbf{i}}^q r_{\mathbf{i}}^{\beta(q)} < r^{-\epsilon}. \tag{4.20}$$

Without loss of generality, we can assume that $0 < r_m < \min\{r_0, \kappa_0\}$ and

$$\frac{\log \frac{r_{\min} r_m}{\operatorname{diam} K_C}}{\log r_{\max}} < r_m^{-\epsilon}.$$

Thanks to Lemma 4.5, we have

$$M_\mu(q, r_m) \le \sum_{j=1}^N p_j^q M_\mu\Big(q, \frac{r_m}{r_j}\Big) + p^q M_\nu(q, r_m) + \lambda_1 r_m^{-\beta(q)}.$$

Recall that

$$\Gamma_1(r_m) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} \le \frac{r_m}{\operatorname{diam} K_C} < r_{\mathbf{i}_-}\Big\},$$
$$\Gamma_2(r_m) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} > \frac{r_m}{\operatorname{diam} K_C}\Big\},$$
$$\Gamma_3(r_m) = \Big\{\mathbf{i} \in \Sigma^* : r_{\mathbf{i}} > \frac{r_m}{\operatorname{diam} C}\Big\}.$$

It is apparent to see that for any $\mathbf{j} \in \Gamma_1(r_m)$,

$$|\mathbf{j}| < \frac{\log \frac{r_{\min} r_m}{\operatorname{diam} K_C}}{\log r_{\max}} < r_m^{-\epsilon}.$$

An easy induction gives that

$$M_\mu(q, r_m) \le \sum_{\mathbf{j}\in\Gamma_1(r_m)} p_{\mathbf{j}}^q M_\mu\Big(q, \frac{r_m}{r_{\mathbf{j}}}\Big) + p^q \sum_{\mathbf{j}\in\Gamma_2(r_m)} p_{\mathbf{j}}^q M_\nu\Big(q, \frac{r_m}{r_{\mathbf{j}}}\Big) + \lambda_1 r_m^{-(\beta(q)+\epsilon)}.$$

For any $\mathbf{j} \in \Gamma_1(r_m)$, it is easy to check that $\frac{r_m}{r_{\mathbf{j}}} \ge \operatorname{diam} K_C$, and in such a case one ball of radius $\frac{r_m}{r_{\mathbf{j}}}$ with center in K_C can cover K_C. It follows that

$$\sum_{\mathbf{j}\in\Gamma_1(r_m)} p_{\mathbf{j}}^q M_\mu\Big(q, \frac{r_m}{r_{\mathbf{j}}}\Big) = \sum_{\mathbf{j}\in\Gamma_1(r_m)} p_{\mathbf{j}}^q \le C_{\max} r_m^{-\beta(q)} \quad \text{(using Lemma 3.2)}.$$

A similar argument also gives that for any $\mathbf{j} \in \Gamma_2(r_m) \setminus \Gamma_3(r_m)$,

$$M_\nu\Big(q, \frac{r_m}{r_{\mathbf{j}}}\Big) = 1.$$

Evidently,

$$\begin{aligned}
\sum_{\mathbf{j}\in\Gamma_2(r_m)\setminus\Gamma_3(r_m)} p_{\mathbf{j}}^q M_\nu\Big(q, \frac{r_m}{r_{\mathbf{j}}}\Big) &= \sum_{\mathbf{j}\in\Gamma_2(r_m)\setminus\Gamma_3(r_m)} p_{\mathbf{j}}^q \\
&\le \max\Big\{\Big(\frac{r_m}{\operatorname{diam} K_C}\Big)^{-\beta(q)}, \Big(\frac{r_m}{\operatorname{diam} C}\Big)^{-\beta(q)}\Big\} \cdot \Big(\sum_{\mathbf{j}\in\Gamma_2(r_m)\setminus\Gamma_3(r_m)} p_{\mathbf{j}}^q r_{\mathbf{j}}^{\beta(q)}\Big) \\
&\le C_{\max} r_m^{-\beta(q)} \cdot \Big(\sum_{\mathbf{j}\in\Gamma_2(r_m)} p_{\mathbf{j}}^q r_{\mathbf{j}}^{\beta(q)}\Big) \\
&\le C_{\max} r_m^{-(\beta(q)+\epsilon)} \quad \text{(using (4.20))}.
\end{aligned}$$

Here we recall that

$$C_{\max} = \max\Big\{(\operatorname{diam} K_C)^{\beta(q)},\ (\operatorname{diam} C)^{\beta(q)},\ \Big(\frac{\operatorname{diam} K_C}{r_{\min}}\Big)^{\beta(q)}\Big\}.$$

Moreover, by the definition of $\dim_A \nu$ in (2.4), there exists $C_1 > 0$ such that for any $\mathbf{j} \in \Gamma_3(r_m)$,

$$\frac{\nu(B(x, \frac{2r_m}{r_{\mathbf{j}}}))}{\nu(B(x, r_m))} \le C_1\Big(\frac{2}{r_{\mathbf{j}}}\Big)^{\dim_A \nu+\epsilon}. \tag{4.21}$$

Thus, for any $\mathbf{j} \in \Gamma_3(r_m)$, we have

$$\begin{aligned} M_\nu\Big(q, \frac{r_m}{r_{\mathbf{j}}}\Big) &\le I_\nu\Big(q, \frac{2r_m}{r_{\mathbf{j}}}\Big) \quad \text{(using Lemma 4.3)} \\ &= \int\limits_C \nu\Big(B\Big(x, \frac{2r_m}{r_{\mathbf{j}}}\Big)\Big)^{q-1}\, d\nu(x) \\ &\le C_1^{q-1}\Big(\frac{2}{r_{\mathbf{j}}}\Big)^{(\dim_A \nu+\epsilon)(q-1)} I_\nu(q, r_m) \quad \text{(using (4.21))} \\ &\le C_1^{q-1}\Big(\frac{2}{r_{\mathbf{j}}}\Big)^{(\dim_A \nu+\epsilon)(q-1)} r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \quad \text{(using 4.19).} \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{\mathbf{j}\in\Gamma_3(r_m)} p_{\mathbf{j}}^q M_\nu\Big(q, \frac{r_m}{r_{\mathbf{j}}}\Big) &\le C_1^{q-1} r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \cdot \Big(\sum_{\mathbf{j}\in\Gamma_3(r_m)} p_{\mathbf{j}}^q \Big(\frac{2}{r_{\mathbf{j}}}\Big)^{(\dim_A \nu+\epsilon)(q-1)}\Big) \\ &\le C_2 r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \cdot \Big(\sum_{\mathbf{j}\in\Gamma_3(r_m)} p_{\mathbf{j}}^q r_{\mathbf{j}}^{(\dim_A \nu+\epsilon)(1-q)}\Big), \end{aligned}$$

where

$$C_2 = 2^{(\dim_A \nu+\epsilon)(q-1)} C_1^{q-1}.$$

Based on the above argument, we conclude that

$$M_\nu(q, r_m) \le C_{\max} r_m^{-\beta(q)} + p^q\Big(C_{\max} r_m^{-(\beta(q)+\epsilon)} + C_2 r_m^{-(\underline{\tau}_\nu(q)+\epsilon)} \sum_{\mathbf{j}\in\Gamma_3(r_m)} p_{\mathbf{j}}^q r_{\mathbf{j}}^{(\dim_A \nu+\epsilon)(1-q)}\Big) + \lambda_1 r_m^{-(\beta(q)+\epsilon)}.$$

This infers that

$$\underline{\tau}_\mu(q) \le \max\{\beta(q) + \epsilon,\ \underline{\tau}_\nu(q) + \epsilon,\ \underline{\tau}_\nu(q) + \epsilon + \beta(q) - (1 - q)(\dim_A \nu + \epsilon)\}.$$

Letting $\epsilon \to 0$ leads to the final result. □

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